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DISSERTATION

COMBAT LOGISTICS PROBLEMS

by

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June 1989

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89 11 20 104

Unclassified

security classification of this page

REPORT DOCUMENTATION PAGE

1a Report Security Classification Unclassified	1b Restrictive Markings
2a Security Classification Authority	3 Distribution Availability of Report
2b Declassification Downgrading Schedule	Approved for public release; distribution is unlimited.
4 Performing Organization Report Number(s)	5 Monitoring Organization Report Number(s)
6a Name of Performing Organization Naval Postgraduate School; 6b Office Symbol (if applicable) 55	7a Name of Monitoring Organization Naval Postgraduate School
6c Address (city, state, and ZIP code) Monterey, CA 93943-5000	7b Address (city, state, and ZIP code) Monterey, CA 93943-5000
8a Name of Funding Sponsoring Organization; 8b Office Symbol (if applicable)	9 Procurement Instrument Identification Number
8c Address (city, state, and ZIP code)	10 Source of Funding Numbers: Program Element No; Project No; Task No; Work Unit Accession No

11 Title (Include security classification) COMBAT LOGISTICS PROBLEMS (Unclassified)

12 Personal Author(s) Steven E. Pilnick

13a Type of Report	13b Time Covered	14 Date of Report (year, month, day)	15 Page Count
Doctoral Dissertation	From To	June 1989	241

16 Supplementary Notation The views expressed in this thesis are those of the author and do not reflect the official policy or position of the Department of Defense or the U.S. Government.

17 Cosati Codes			18 Subject Terms (continue on reverse if necessary and identify by block number)
Field	Group	Subgroup	operational logistics, stochastic scheduling, dynamic programming,queueing,diffusion approximation,transient analysis

19 Abstract (continue on reverse if necessary and identify by block number)

Models are developed for two different Combat Logistics situations, one in the area of Operational Combat Logistics and the other in Combat Support Logistics. In the first situation, Operational Combat Logistics models are developed to assist in scheduling the replenishment of weapons within a Navy Battle Group following a combat engagement. Consideration is given to the uncertain arrival of a follow-on attack which may interrupt the replenishment process before all requirements are satisfied. In a justifiably simplified approach, optimal Vertical Replenishment scheduling is achieved by sequencing lifts in decreasing order of an index, called Logistics Weighted Combat Value (LWCV). The ~~LWCV method~~ is then used in an ~~efficient scheduling heuristic for a realistic model~~ and produces results which compare very favorably with a locally optimum schedule obtained with a lengthy local neighborhood ~~search.~~ Separately, for a simple model, optimal Connected Replenishment scheduling is achieved with dynamic programming (DP). The DP approach is then adapted to more realistic situations. ~~Examples of the implementations of these methods are presented.~~ In the second situation, Combat Support Logistics models are developed to analyze the combat availability of a system supported by a single diagnosis repair test facility. A characteristic that distinguishes Combat Support Logistics from peacetime in-service support, is that in peacetime, a logistics system may operate in steady-state, whereas, because of the dynamic intensity of combat, steady-state conditions may never be reached in periods of conflict. The modeling technique is to use a diffusion approximation valid for the heavy traffic conditions anticipated under combat conditions. The ~~simple analytic solutions obtained~~ are compared to simulation results and found to be very satisfactory. ~~Alternative scheduling policies~~ that reflect different organizational ~~maintenance~~ service disci-~~plines can be readily compared.~~ ~~The model provides a framework for choosing near-optimal spare module allocations within~~ budget constraints.

20 Distribution Availability of Abstract ☒ unclassified unlimited ☐ same as report ☐ DTIC users	21 Abstract Security Classification Unclassified

22a Name of Responsible Individual	22b Telephone (include Area code)	22c Office Symbol
Donald P. Gaver	(408) 646-2605	code 55Gv

DD FORM 1473,84 MAR

83 APR edition may be used until exhausted
All other editions are obsolete

security classification of this page

Unclassified

Combat Logistics Problems

by

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Submitted in partial fulfillment of the
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DOCTOR OF PHILOSOPHY IN OPERATIONS RESEARCH

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ABSTRACT

Models are developed for two different Combat Logistics situations, one in the area of Operational Combat Logistics and the other in Combat Support Logistics. In the first situation, Operational Combat Logistics models are developed to assist in scheduling the replenishment of weapons within a Navy Battle Group following a combat engagement. Consideration is given to the uncertain arrival of a follow-on attack which may interrupt the replenishment process before all requirements are satisfied. In a justifiably simplified approach, optimal Vertical Replenishment scheduling is achieved by sequencing lifts in decreasing order of an index, called Logistics Weighted Combat Value (LWCV). The LWCV method is then used in an efficient scheduling heuristic for a realistic model and produces results which compare very favorably with a locally optimum schedule obtained with a lengthy local neighborhood search. Separately, for a simple model, optimal Connected Replenishment scheduling is achieved with dynamic programming (DP). The DP approach is then adapted to more realistic situations. Examples of the implementations of these methods are presented. In the second situation, Combat Support Logistics models are developed to analyze the combat availability of a system supported by a single diagnosis repair test facility. A characteristic that distinguishes Combat Support Logistics from peacetime in-service support, is that in peacetime, a logistics system may operate in steady-state, whereas, because of the dynamic intensity of combat, steady-state conditions may never be reached in periods of conflict. The modeling technique is to use a diffusion approximation valid for the heavy traffic conditions anticipated under combat conditions. The simple analytic solutions obtained are compared to simulation results and found to be very satisfactory. Alternative scheduling policies that reflect different organizational maintenance service disciplines can be readily compared. The model provides a framework for choosing near-optimal spare module allocations within budget constraints.

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ACKNOWLEDGEMENTS

The completion of this thesis marks the end of a very challenging and rewarding period in my life. The U. S. Navy gave me the unique opportunity to enter a program of doctoral studies and research at the Naval Postgraduate School, and for that I am grateful. I am particularly grateful that my primary career in the Surface Warfare community of the Navy did not run aground while I was navigating the restricted waters of academia, and that I am being permitted to get back out into blue waters.

Thanks are due to all the members of my committee and the faculty of the Department of Operations Research who gave of themselves -- teaching, encouraging, and supporting -- to enable me to complete the program.

Most of all, I want to thank Professor Donald Gaver who, as my advisor, allowed me to work in a research environment of great independence while providing just the right amount of supervision to keep me on course. He taught me more than just the mechanics of stochastic modeling; he imparted a feeling for applying the tools of the trade to real problems. His guidance and support were instrumental to my completion of the program.

A special acknowledgement also goes to Professor Gerald Brown, who not only served on my doctoral committee, but, as Chairman of the OR Department PhD Committee, looked after me during my entire stay at the Postgraduate School.

I would also like to thank Professor Patricia Jacobs for her careful reading of the early drafts of the entire thesis. Her suggestions led to substantial improvements in the final product.

Thanks are also due to Professor John Lehoczky of Carnegie-Mellon University who reviewed parts of the thesis and provided helpful comments early in my program, and to Professor David Schrady, and Commanders Mark Mitchell and David Wadsworth who provided much support in the area of Operational Logistics. I would also like to specifically thank Professors Alan Washburn and Dan Boger, each of whom read through an entire draft and provided suggestions that improved the finished copy.

A special word of thanks goes to CDR Michael Olson, with whom I went through the PhD program. His support through all stages of our studies, exams, and research were invaluable. I can not imagine what it would have been like without him.

At last, I must acknowledge the support I received from my family. My greatest thanks go to my wife, Mary Lou, who not only assumed the burden of home and family responsibilities while I was absorbed in my studies and research, but also took on extra responsibilities as the President of the Officer Students' Wives' Club, as well as other volunteer positions, where she demonstrated serious personal involvement and dedication concerning *our* career. I thank my children; David, who was very understanding of the endless hours that Daddy had to spend at "weekend school" and "night school", and who provided the occasional distractions that were needed to keep my life in balance; and Michael, who by his own continued excellence in school and outside activities, enabled me to pursue my studies and research without feeling guilty about not being around more. Finally, I thank my parents, who were never hesitant with their encouragement. I also appreciate that, despite the challenge, they have made a reasonable effort to try to understand what Operations Research is.

I. COMBAT LOGISTICS OVERVIEW

A sound logistic plan is the foundation upon which a war operation should be based. If the necessary minimum of logistics support can not be given to the combatant forces involved, the operation may fail, or at least be only partially successful.

-- R. A. Spruance

A. NAVY LOGISTICS BACKGROUND

It is beyond the scope of this work to provide a comprehensive survey of logistics in the Navy. However, some concepts are briefly described here to establish a frame of reference.

As defined by the Chief of Naval Operations [Ref. 1], the Navy Logistics System comprises three primary, interacting functions: acquisition logistics, in-service support, and operational logistics.

1. Operational Logistics in Combat

Operational logistics concerns the allocation of logistics support resources *at all levels* within the Operating Forces to enable the successful execution of assigned missions. One of the levels specified is Battle Force/Unit Logistics, which includes the planning, management and execution of logistics activities within the Battle Force or Unit.

The ships whose primary mission it is to conduct Battle Force/Unit logistics activities are the supply, ammunition, and fuel replenishment ships which are collectively referred to as Combat Logistics Force ships. The term *Combat Logistics Force* is fairly new. The previous terminology was *Mobile Logistics Support Force* and before that *Service Force*. Although the term *combat logistics* is used in this context, it is not explicitly defined in the Navy literature. The current terminology is generally taken as a reflection of the operational potential of the CLF ships to deploy as a part of a Battle Group, or otherwise directly support a Battle Group, whether or not that Battle Group actually engages combat, or otherwise faces imminent attack. In this current work, the term combat logistics is used more specifically to reflect a direct association with actual

combat. The following definition is adopted to describe operational combat logistics as a specialization of operational logistics:

Definition 1.1: Operational Combat Logistics comprises logistics activities which are conducted within combatant forces, during an ongoing or imminent combat, and which directly affect the outcome of the combat.

A specific logistics function that clearly belongs in the area of operational combat logistics is that of resupplying ammunition to combatant ships during the interval between successive raids of attacking aircraft. This type of combat logistics is the focus of the next three chapters. Chapter II provides an introduction to the problem of replenishing Battle Group ammunition during combat, and the models of Chapters III and IV deal with aspects of this problem.

2. In-service Support of Combat Operations

In-service support concerns the distribution of necessary supplies and proper maintenance of weapon and support systems to ensure that peacetime and wartime Navy readiness and sustainability goals are met. One of the principal in-service support functions performed by elements of the Navy shore establishment and operating forces is called simply Navy Maintenance.

The actual maintenance of Navy ships, aircraft, submarines, weapons and equipment is performed on a highly decentralized basis within the various Navy communities by fleet units, contractors, depots and shipyards. Whereas all levels of maintenance support are concerned, ultimately, with returning the serviced unit to a condition in which it can carry out its mission, including combat, organizational level maintenance by the repair personnel of a deployed unit can be the most closely related to sustaining combat operations. The following definition is adopted to describe combat support logistics as a specialization of in-service support:

Definition 1.2: Combat support logistics comprises the supply and distribution of vital weapon systems components, and corrective maintenance of weapon systems within combatant forces, during an ongoing or imminent combat, and which directly affect the sustainability of combat operations.

A characteristic that distinguishes combat support logistics from peacetime in-service support, is that in peacetime, a logistics system may operate in steady-state, whereas due to the dynamic intensity of combat, steady-state conditions may never be reached in periods of conflict. The models of Chapter V relate to a problem in combat support logistics -- the transient analysis of the effect of alternative repair service policies on combat system availability.

II. INTRODUCTION TO THE PROBLEM OF REPLENISHING AMMUNITION DURING COMBAT

A. THE COMBAT REPLENISHMENT PROBLEM

The scenario in which the problem being studied may arise is set in a conventional hot war. A Carrier Battle Group is operating in an area where it is subject to attack by enemy aircraft with anti-ship missiles. It is anticipated that air raids will occur in large waves. The time between waves is available for replenishing anti-aircraft ammunition within the Battle Group in anticipation of the next raid. Replenishment ammunition is stocked by an on-station ammunition ship that can provide limited parallel service. However, the time between waves is uncertain and likely to be insufficient to satisfy all requirements. Besides limited time available, the quantity of ammunition available from the Battle Group on-station replenishment ship may be less than the total requirements. The problem facing the decision maker may be simply stated: how *best* to replenish ammunition in the uncertain time available between raids?

B. OPERATIONAL LOGISTICS BACKGROUND

1. Afloat Logistics

Replenishment at sea is conducted from combat logistics force ships designed for that purpose. Ships that provide mainly a single commodity include the ships designated as AE, AO, and AFS, which carry ammunition, fuel, or stores respectively. Multi-product replenishment ships include the AOE and AOR which carry a mix of fuel and ammunition.

Hereford and Spiegel [Ref. 2] gave the following unclassified description of what they refer to as the U. S. Navy's afloat logistics system.

> In order to maximize the utility of units, the U. S. Navy has developed a system to resupply carrier battle forces while they are at sea. (Figure 1) shows a schematic representation of the Navy's primary means of providing afloat logistics support in wartime. Supplies (POL, ordnance, stores, spare parts, etc.) are brought by strategic lift assets to advanced support bases. Here they are transferred to console ships. The console ships are single product ships (oilers, ammunition ships, stores ships) that then bring the supplies to the operations areas of the battle force.

Once at the battle force, the supplies are transferred to a multiproduct station ship (AOE) that then is charged with redistributing the supplies to the other ships in the battle force. Constituents of the battle force can also be serviced by single product shuttle ships. The ideal station ships (such as the present AOEs) have sufficient speed to maintain position with the battle force at all times.

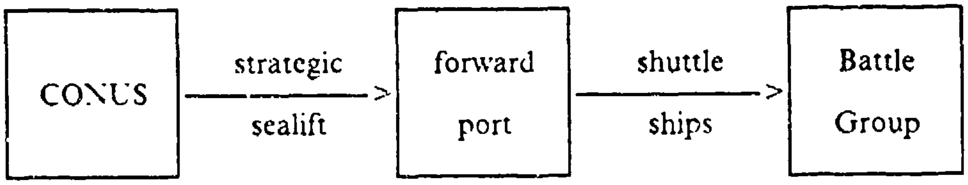


Figure 1. Model of Underway Replenishment Process

The problem of replenishment at sea during combat is complicated by the possibility that while in formation for combat, anti-air warfare ships may be dispersed at great distances from the formation center where the carrier and logistics ship are likely to be. To illustrate the expanse of a modern Carrier Battle Group dispersed formation, then Chief of Naval Operations, Admiral Watkins [Ref. 3] overlaid a Battle Group on a map of the east coast of the United States to show that with the center of the formation located in Washington, DC, anti-air warfare ships might be stationed in Philadelphia, Pa., Harrisburg, Pa, Clarksburg, W.Va., Norfolk, Va., Trenton, N.J., and Dover, Del.

2. Underway Replenishment of Ammunition

There are two basic methods for a station ship to physically transfer ammunition at sea to a combatant ship, which are described in the doctrinal naval warfare publication NWP 14 [Ref. 4]. One method is called *connected replenishment*, or *CONREP* for short, and the other method which uses helicopters is called *vertical replenishment*, or *VERTREP* for short. In both methods the AE, AOE, or AOR that provides the ammunition is referred to as the *delivery ship*, and the *combatant* unit that is to be serviced (replenished) is referred to as the *receiving ship*.

For CONREP, the delivery ship maintains a steady course at moderate speed, and the receiving ship maneuvers into a position parallel to the delivery ship and separated by about 30 meters. While *alongside*, one or more wire highlines are rigged between the two ships, and pallets or containers of ammunition are winched from the delivery ship to the receiving ship.

For VERTREP, a logistics helicopter is used to lift the pallets or containers of ammunition from a pickup area on the delivery ship to a drop area on the receiving ship. VERTREP may be conducted concurrently with CONREP while the receiving ship is alongside, or at greater distance. Typically, the distance is kept close to maintain a high transfer rate, but VERTREP could be conducted at greater ranges limited primarily by command and control considerations.

Prior to transfer by either CONREP or VERTREP, the delivery ship must remove ammunition from storage magazines and stage it at the delivery station. These delivery ship replenishment activities will be collectively referred to as *breakout*. After transfer by either CONREP or VERTREP, the receiving ship must move the ammunition from the receiving station and load it into the appropriate magazine. These receiving ship replenishment activities will be collectively referred to as *strikedown*. Also, depending on the particular type of weapon, breakout and strikedown activities may include changing ordnance from a storage configuration to a transfer configuration, and then changing it from a transfer configuration to ready-for-use configuration, respectively.

Stiles [Ref. 5] provided an unclassified discussion of how significant strikedown time can be in the case of reloading the most modern missile launcher, the Mk-41 Vertical Launching System (VLS), which is installed in the most capable anti-air warfare ships in the U. S. Navy, the AEGIS cruisers.

> The greatest limiting factor in terms of both speed and flexibility of VLS UnRep is the strikedown crane and the assorted deck-handling equipment used in conjunction with it. UnRep ships are currently capable of passing over many more missiles than the VLS crane is capable of striking down.

Anderberg, Feldman and Odell [Ref. 6] raised the following questions, which remain unanswered, concerning ordnance replenishment in their analysis of operational logistics in a major fleet exercise:

- How do the (decision makers) decide on the precedence of one ordnance replenishment over another?
- Given the anticipated length of time required for fully rearming a guided-missile destroyer or cruiser, how does time alongside get rationed among ships needing replenishment so that all of the time is not consumed by one ship?

The models in the following chapters are aimed at answering these questions. Chapter III considers scheduling of VERTREP, and Chapter IV models the CONREP problem.

C. MODEL FORMULATION PRELIMINARIES

1. Measures of Effectiveness and Objectives

The concept of *combat logistics* manifests itself in identifying objectives and appropriate measures of effectiveness for the problem. If combat outcome were not considered, then objectives could be stated in *pure* logistics terms. Examples of pure logistics objectives are given in Figure 2.

Maximize the (..........) number of .... of weapons transferred.

expected	rounds
minimum	tons
	lifts

Minimize the (..........) time to transfer .... of the weapons requested.

expected	all
minimum	some number
maximum	some percentage
variance of	

Maximize the probability that a particular level of re-arming is completed by a deadline.

Minimize the time ships are away from assigned stations for replenishment.

Figure 2. Examples of Pure Logistics Objectives

If logistics activities were not considered, then objectives could be stated in pure combat terms. Examples of *pure combat* objectives are given in Figure 3.

Maximize the (..........) number of enemy ....
expected platforms engaged
minimum ASMs destroyed

Maximize the (..........) number of own .... surviving.
expected ships
minimum tonnage
aircraft
people

Maximize the (..........) number of successive waves survived.
expected
minimum

Minimize the (..........) number of enemy .... that penetrate defenses.
expected platforms
maximum ASMs

Minimize the (..........) number of own .... lost.
expected ships
maximum tonnage
aircraft
people

Maximize the probability of survival of some number of own forces.

Maximize the probability of kill of some number of enemy forces.

Figure 3. Examples of Pure Combat Objectives

There would be several deficiencies in the results of the modeling if combat and logistics were considered separately. At one extreme, if time to conduct transfers (a pure logistics consideration) were not considered, and the only criterion was which weapons are most important regardless of the time it takes to transfer them, then a clearly undesirable result could be that the entire time available could be consumed (slowly) transferring a few Vertical Launch missiles. At the other extreme, if time to conduct transfers were the only consideration, disregarding the combat value of weapons, then another undesirable result could be that the entire time available could naively be allotted to only making VERTREP transfers to receivers who were at minimum range so as to maximize transfer rates, without considering that by taking a little more time, much more combat value may accrue.

Combat Logistics objectives can be thought of as: a fusion of *pure* logistics objectives and *pure* combat objectives. They could be thought of as combat objectives expressed as functions of a logistics process, or logistics objectives "weighted" by the value of the material in combat. Examples of *combat logistics* objectives are given in Figure 4.

Maximize the expected additional enemy kills due to weapons transferred.
Minimize the maximum time required to transfer those weapons that provide some specified probability of mission success.
Maximize the expected total *combat value* of weapons transferred.

Figure 4. Examples of Combat Logistics Objectives

In this work, models are developed that seek to maximize the expected *combat value* of weapons transfer completions prior to the next raid arrival. A very simple combat model is used in Chapter III to quantify the idea of *combat value* in a particular combat scenario. That simple combat model is subsequently examined to provide insight into the characteristics that should be captured in a more general combat value function, and a heuristic method to derive combat values is proposed in an appendix.

2. Units of Measurement for Weapons

It is convenient to specify what units should be used to count numbers of weapons. As with the choice of measures of effectiveness, there are several possibilities in the context of a combat logistics problem.

At one extreme, in some *pure* combat models an appropriate unit of measurement for weapons might be a *round* of ammunition, such as *a* missile. However, if it is desired to consider different weapons, the modeling would encounter order of magnitude differences if comparing, say, one surface-to-air missile round, with one round of antiaircraft gun ammunition. Besides confounding combat effectiveness comparisons, these *scale* differences would be especially pronounced in logistics, where individual rounds of ammunition may vary greatly in weight and volume.

At the other extreme, in some *pure* logistics models an appropriate unit of measurement for weapons might be a *ton* of ammunition. This has the advantage of overcoming some of the problems of scale, and may be particularly appropriate and useful in a model concerning sealift or airlift. It is, however, not an *operational* unit of measurement, readily used by the combatant ships who receive the weapons.

Between these two extremes, there is the operational logistics problem of Battle Group replenishment. Here, it is suggested that the *natural* unit of measurement for weapons is a lift of ammunition. On the logistics side, a lift is the unit that is actually handled by rig crews, helos, dollies, forklifts, etc. And with respect to combat, a lift aggregates *smaller* ordnance items, like rounds of gun ammunition and chaff, so that the units are comparable with respect to combat effectiveness (i.e., it is not sensible to compare one round of 76mm gun ammunition with one Standard surface-to-air missile; it is more reasonable to compare one missile with one pallet of 76mm.)

3. Weapons State

Using common military terminology, the number of weapons available for combat is referred to as a *weapons state.* The weapons state of the entire battle group may be thought of as a vector of the weapons states of the individual ships in the battle group. To consider more detail, an individual ship's weapons state may itself be a vector of the weapons state of each type of weapon carried.

D. OPERATIONS RESEARCH BACKGROUND

It appears that the most closely related operations research models for this problem are in the area of *scheduling theory* in general, and *stochastic shop scheduling* in particular. A brief review of the pertinent terminology and literature follows.

1. Scheduling Theory Terminology

The terminology of scheduling theory comes from the manufacturing industry; see Conway, Maxwell, and Miller [Ref. 7] or French [Ref. 8]. Most authors use the idea of scheduling some number of *jobs* to be processed through some number of *machines.* In the *general job-shop problem*, each job has its own processing order that may be unrelated to the processing order of other jobs. A special case of a job-shop which occurs when all the jobs have the same processing order is referred to as a *flow-shop*, because

the jobs flow between machines in the same order. The processing of a job on a machine is called an *operation*, and the length of time it takes to perform an operation is called the *processing time*. Typically, the time required to *set up* a machine to process a job is included in the processing time. The epoch at which an operation ends is called the *completion time*. If an operation is required to be completed by a particular time, that time is called a *due date*. Some common measures of effectiveness in scheduling theory relate completion times and due dates. *Lateness* of a job is the difference between its completion time and due date. A positive difference is called *tardiness*, and a negative difference is called *earliness*. The *number of late jobs*, or *number of tardy jobs* counts the number of jobs where completion time exceeds the due date. The contribution of each job to any of these measures of effectiveness may be *weighted* by the relative importance of the job. The time at which a job becomes available for processing is called its *release date*. If the number of jobs and their release dates are known and fixed, the problem is said to have a *static arrival process*. In contrast, if the jobs arrive randomly, the problem is said to have a *dynamic arrival process*.

In the Battle Group Ammunition Replenishment problem, the lifts are jobs; the breakout on the delivery ship, transfer via CONREP station or VERTREP helicopter, and strikedown on the receiving ship are operations on machines; and the jobs must follow the path of *breakout machine* to *transfer machine* to *strikedown machine* which defines a flow-shop. The breakout, transfer, and strikedown times are processing times. All lift requirements are known at the outset which defines a static arrival process for jobs; and the time at which each receiver can receive his first lift is a release date. The time by which strikedown of the lifts must be completed so that the ordnance is available for combat is a due date.

2. Stochastic Considerations

Stochastic considerations enter shop scheduling problems in the literature in several ways. The most common is in the form of stochastic processing times. Another form is stochastic release dates or due dates; see Pinedo and Schrage [Ref. 9], Pinedo [Ref. 10], Coffman [Ref. 11], and Dempster, Lenstra, and Rinnooy Kan [Ref. 12].

The dominant stochastic element in the Battle Group Ammunition Replenishment problem is the time of arrival of the next wave of attack which may be thought of as a stochastic due date. Coping with the uncertainty of a raid's arrival time, and immediate replenishment termination is clearly of utmost importance in this setting.

In addition to the stochastic due date, there is some inherent variability in the release dates and processing times. In the case of release dates, however, the time at which each receiver can receive his first lift is mostly determined by the relative positions of the delivery ship and the receiving ship, and the relative speeds at which they maneuver, or relative speed at which a VERTREP helicopter flies between them. Since those positions and speeds (which may depend on the current wind and weather conditions) are generally known at the outset of a replenishment period, release dates will be treated as deterministic. Similarly, in the case of processing times, the attributes of each job are known. Each job is a particular lift of ordnance, the operations to process each lift are known, and times to perform those operations under normal circumstances are known, at least approximately. To obtain an initial schedule, processing times under normal circumstances will be treated as deterministic.

Another area in which uncertainty enters the problem concerns random equipment breakdowns. If a fixed schedule were to be developed and strictly adhered to, random breakdowns could seriously impact the objectives of the replenishment. However, the replenishment process is continually observed, and it is known when a breakdown occurs. When that happens, if an estimated time of repair (possibly infinite) can be given, then the initial schedule can be revised. The issue of random equipment breakdowns will be handled by *dynamic schedule revision.*

The objective of the Battle Group Ammunition Replenishment problem, which was described above as maximizing the expected combat value of weapons transfer completions prior to the next raid arrival, may be expressed in scheduling theory terminology and is equivalent to *minimizing the weighted expected number of late jobs in a flow shop with a stochastic due date.* The weights in this case are the combat values.

3. Dynamic and Stochastic Scheduling

For stochastic scheduling problems with a dynamic arrival process, the predominant theoretical approach is that of queueing theory; see Conway, Maxwell, and Miller [Ref. 7] for a treatment of the interrelation between queueing theory and stochastic scheduling. In the replenishment problem, however, the arrival process is static, so that other approaches arise.

A theoretical approach to stochastic scheduling that is quite distinct from queueing theory has been developed by Gittens and others; see Gittens and Nash [Ref. 13] and Gittens [Ref. 14]. This method assigns a *dynamic allocation index* (also referred

to as a Gittens index) to each job, and then schedules the jobs in decreasing order of this index. The Gittens index is updated as the jobs are processed (hence the word *dynamic*) which allows the schedule to adapt to the realization of arrivals and processing times.

In Chapter III, a prototype model is developed for scheduling VERTREP under special conditions, which leads to scheduling lifts in decreasing order of a ratio, which in the context of the replenishment problem is called a *logistics weighted combat value* (LWCV). Although the development of the model does not invoke the Gittens approach -- it uses an interchange argument (see Ross [Ref. 15]) -- the result corresponds to what Gittens [Ref. 16] calls *forwards induction*, and the LWCV ratio is an example of a Gittens index. The simple VERTREP problem considered in the prototype model is thus an example of a problem for which the use of a Gittens Index will produce an optimal schedule.

A more general methodology that may be applied to stochastic scheduling problems uses *backward induction*, so as to take into consideration future rewards as well as immediate rewards. In the CONREP model of Chapter IV, the principal approach is dynamic programming; see Bellman [Ref. 17], Denardo [Ref. 18], Minoux [Ref. 19], Ross [Ref. 15], or Whittle [Refs. 20, 21].

III. THE COMBAT VERTREP PROBLEM

The vertical replenishment (VERTREP) problem is to determine the best sequence in which to dispatch replenishment ammunition by helicopter from a delivery ship to several receiving ships. When an attack is anticipated, the problem is called the Combat VERTREP Problem, and the sequence of transfers should be *best* with respect to a Combat Logistics objective, as discussed in the previous chapter.

A. A PROTOTYPE MODEL

A prototype model is developed in which an optimal sequence of deliveries can be determined by criteria due to an interchange argument. This is an extension of a model given by Ross [Ref. 15]. The basic idea in the interchange argument is that an arbitrary sequence of deliveries is considered, and then another sequence is determined by interchanging any two consecutive deliveries. The conditions under which this interchange leads to an improvement in the measure of effectiveness are then examined.

1. A Simple VERTREP Problem

This prototype model considers the problem of scheduling VERTREP deliveries with one delivery helicopter within a Battle Group which contains one delivery ship and several receiving ships. Each receiver requests several deliveries, or *lifts*, of ammunition. The time available to conduct ammunition transfers, the air raid interarrival time, is a random variable, with known distribution. The times to conduct transfers to each receiver, which include helicopter delivery times and receiver strikedown times are assumed to be known. When an air raid arrives, it terminates the replenishment process; transfers in progress are not completed. In this model, let the value of having some specified number of weapons available when combat commences be quantified by a measure called *combat value*. Since the time when the replenishment process terminates (and combat commences) is uncertain, the measure of effectiveness to be maximized by choice of delivery sequence is *expected combat value*. Let T denote the air raid arrival time.

2. The Replenishment Process

This replenishment process is now described under an arbitrary ordering of deliveries. Let L denote the total number of lifts to be sequenced, and the index l denote the sequence in which lifts are delivered; $l \in \{1, \dots, L\}$. Let the variable D_l denote the known time it takes a helicopter to pick up the l^{th} lift from the delivery ship, fly to the receiving ship, and drop off the lift; let R_l, denote the known helicopter return time after dropping off the lift; and let S_l denote the time it takes the receiver from when the lift is dropped off until strikedown is complete. The replenishment process, as depicted in Figure 5, starts with the delivery helicopter departing the delivery ship with the first lift at time $t = 0$. It takes time D_1 to deliver the first lift. Then, at time $t = D_1$, the receiver of the first lift immediately starts strikedown, and the helo returns for the second lift. It is assumed that strikedown queues do not develop on the receivers, and as a consequence, delivery or strikedown of subsequent lifts are not precluded or delayed by lifts previously delivered (i.e., there is no *blocking*), and strikedown completions are in the same order as deliveries. One way to model this is to assume that $S_l \leq R_l$, for all lifts.[1] The first transfer is completed at time $t = D_1 + S_1$; the helo returns from the first lift and picks up the second lift at time $t = D_1 + R_1$; the second transfer is completed at time $t = D_1 + R_1 + S_2$; etc.

Let the variable, V_l, denote the total combat value of all weapons available after completion of the l^{th} transfer; and V_0, denote the combat value of weapons initially available before replenishment. Then, define the *marginal* value of the l^{th} lift as $v_l = V_l - V_{l-1}$; where the use of lower case represents marginal, or incremental, change in combat value. Assume the marginal values are non-negative. The accumulation of total combat value during the replenishment process is shown in Figure 6.

3. Expected Combat Value

An expression is now derived for the expected combat value under an arbitrary ordering of deliveries. Start by observing that the total combat value attained under any ordering equals V_0 if an attack arrives and interrupts the replenishment process before completion of the first transfer. It equals V_1 if an attack arrives after completion of the first transfer and before completion of the second transfer. And, in general, the total combat value attained under any ordering equals V_l if an attack arrives after completion of the l^{th} transfer and before completion of the $l + 1^{st}$ transfer, for $l = 1, \dots, L - 1$.

---

1 Less restrictive conditions are discussed later.

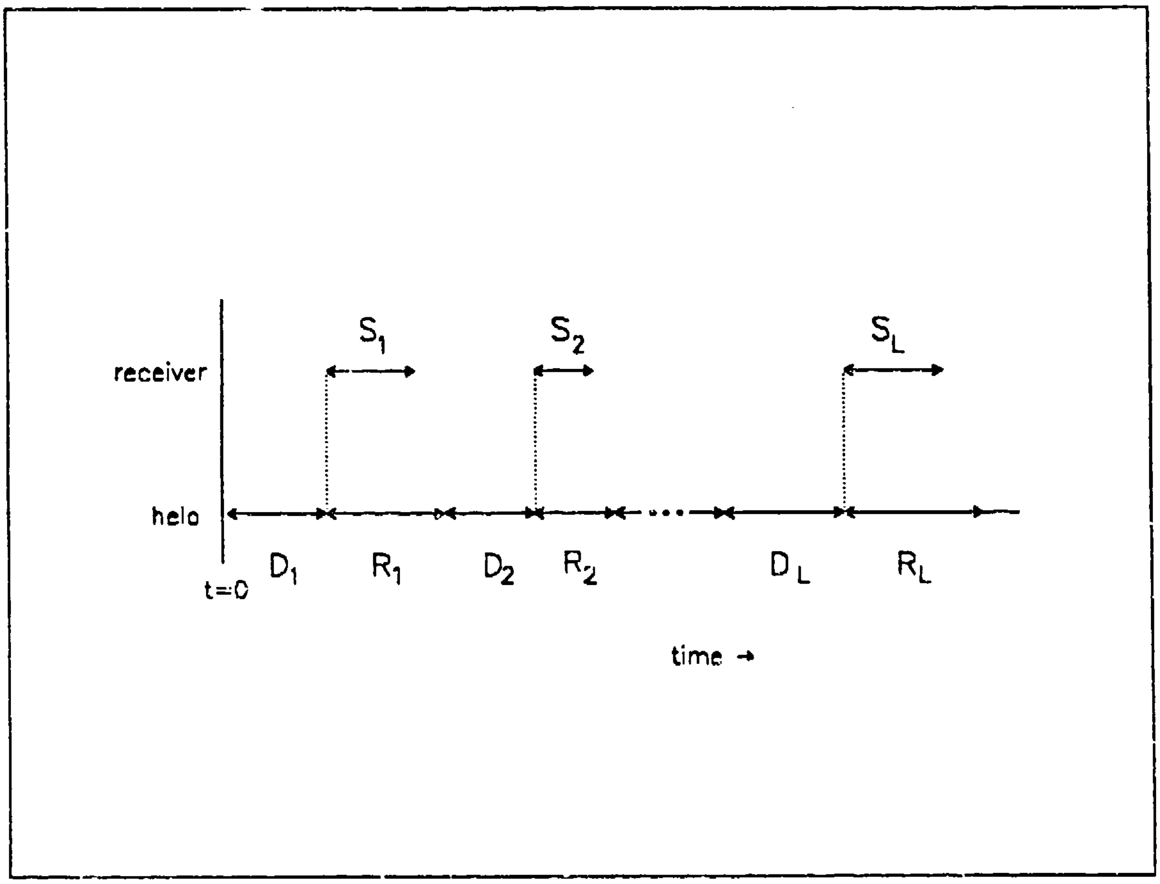


Figure 5. Replenishment Process

Finally, it equals V_L if an attack does not arrive until after completion of the last transfer.

To write an expression for this, let O_* denote any arbitrary ordering, where the subscript will be used later to distinguish specific orderings. Let $V(O_*)$ denote the total combat value under ordering O_*, which can then be expressed

$$\begin{aligned} V(O_*) &= V_0 \text{ if } [D_1 + S_1 > T] \\ &= V_1 \text{ if } [D_1 + S_1 \le T] \wedge [D_2 + S_2 + D_1 + R_1 > T] \\ &= V_2 \text{ if } [D_2 + S_2 + D_1 + R_1 \le T] \wedge [D_3 + S_3 + D_1 + R_1 + D_2 + R_2 > T] \\ &\vdots \\ &= V_L \text{ if } [D_L + S_L + \sum_{l=1}^{L-1} (D_l + R_l) \le T] \ . \end{aligned} \tag{3.1}$$

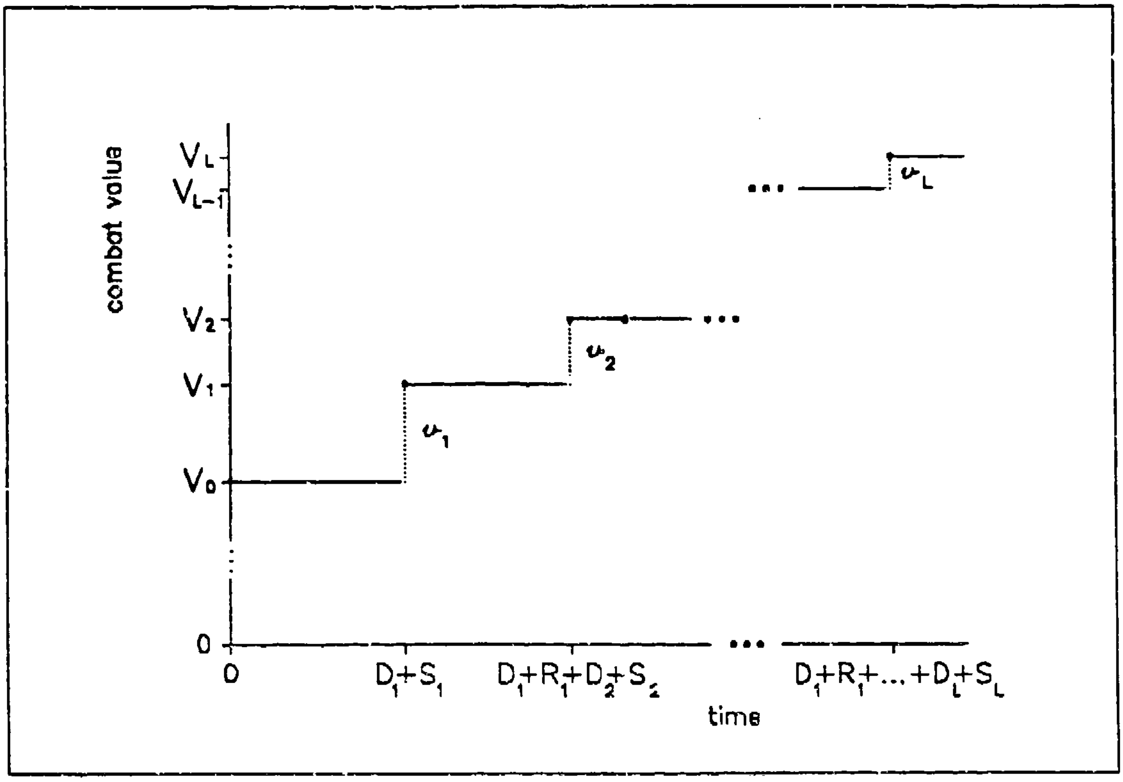


Figure 6. Combat Value

For compactness, let the variable W_k denote the partial sum of delivery and return times through the k^{th} lift; that is

$$W_k = \sum_{l=1}^{k} (D_l + R_l) \quad .$$

If each V_l is then expanded as a sum of the initial value and marginal increases, then (3.1) can be rewritten as

$$\begin{aligned} V(O_\bullet) &= V_0 && \text{if } [T < D_1 + S_1] \\ &= V_0 + v_1 && \text{if } [D_1 + S_1 \le T < D_2 + S_2 + W_1] \\ &= V_0 + v_1 + v_2 && \text{if } [D_2 + S_2 + W_1 \le T < D_3 + S_3 + W_2] \\ &\vdots \\ &= V_0 + v_1 + v_2 + \ldots + v_L && \text{if } [D_L + S_L + W_{L-1} \le T] \quad . \end{aligned} \tag{3.2}$$

Using the representation of (3.2), the expected value can be expressed as

$$
\begin{aligned}
\mathrm{E}[V(O_\bullet)] = V_0 &+ v_1 \ \mathrm{P}[T \geq D_1 + S_1] \\
&+ v_2 \ \mathrm{P}[T \geq D_2 + S_2 + W_1] \\
&+ \dots + v_L \ \mathrm{P}[T \geq D_L + S_L + W_{L-1}] \ .
\end{aligned}
\tag{3.3}
$$

4. Given Data

a. Battle Group Replenishment Parameters

There are M receivers. Let the index r identify a receiver; $r \in \{1, \dots, M\}$. Receiver r requests n_r lifts of ammunition. The total number of lifts to be sequenced is

$$
L = \sum_{r=1}^{M} n_r \ .
$$

Lifts are pre-staged on the delivery ship in the sequence (determined by the model) so that breakout time at the delivery ship does enter this model. Helicopter pickup and drop-off handling times and flight speeds with and without loads are known. Each receiver is on an assigned (fixed) station for combat and replenishment, at known bearing and range from the delivery ship, and on formation course and speed. Given this data, delivery time to each receiver, δ_r , and return flight time from each receiver, ρ_r , are determined. Thus, if the l^{th} lift is delivered to receiver r , then $D_l = \delta_r$; and $R_l = \rho_r$. The total round-trip shuttle time between the delivery ship and each receiver, is $\delta_r + \rho_r$.

Strikedown times per lift for each receiver, ψ_r, are known. If the l^{th} lift is delivered to receiver r , then $S_l = \psi_r$. To be consistent with the assumption that $S_l \leq R_l$, for all l , it is also assumed that $\psi_r \leq \rho_r$, for all r. Thus the following assumptions have been made:

Assumption 3.1: Helicopter delivery and return times, and strikedown times for each receiver are fixed constants.

Assumption 3.2: To preclude the development of strikedown queues, $\psi_r \leq \rho_r$, for all r .

The next assumption concerns the air raid interarrival time.

Assumption 3.3: Air raid interarrival time is a random variable, T, assumed to have an exponential distribution with a known mean, τ .

b. Battle Group Combat Value Function

Here, a simple combat model is developed to derive a candidate for a given combat value function for the Battle Group. Further, an expression is derived for the marginal increase in combat value due to unit increases in the number of weapons available for combat in the Battle Group.

Combat Model. In this combat model the Battle Group is defended with several defenders from attack by a single bomber. The defenders are each of the ships in the Battle Group that, during replenishment, are referred to as the receivers. As above, the defenders receivers are indexed by $r \in \{1, \dots, M\}$. Each defender has one anti-aircraft missile system and several missiles. The number of missiles on each defender available for combat is called their *missile state*, denoted by s_r. The missile state of the entire Battle Group is the vector of individual missile states: $(s_1, \dots, s_M)$. The defense fails if all missiles in the Battle Group fail to kill the attacker. The single shot kill probability of the missile system on defender r, denoted by p_r ; and the probability that the attacker is engageable by defender r, denoted by π_r ; are given. For this simple model, assume that engageability by defenders is mutually exclusive (no overlap), and let π_0 denote the probability that the attacker is not engageable by any defender. Then,

$$\sum_{r=0}^{M} \pi_r = 1 \quad .$$

Combat Value Function. The measure of combat value (sometimes called a measure of combat effectiveness or *utility*) of the weapons available, is taken to be the probability that the single attacker is shot down, expressed as a function of the missile state of the Battle Group. That is, define

$$U(s_1, \dots, s_M) = \mathrm{P}\,[\text{Attacker killed}] \quad .$$

In terms of given parameters, U can be written

$$U(s_1, \dots, s_M) = 1 - \pi_0 - \sum_{r=1}^{M} \pi_r (1 - p_r)^{s_r} \quad . \tag{3.4}$$

Marginal Combat Value. An expression is now derived for the marginal combat value of transferring one additional missile to receiver r, whose current missile state is s_r. Using lower case to represent *marginal* value, define

$$u_r(s_r) = U(s_1, \dots, s_r + 1, \dots, s_M) - U(s_1, \dots, s_r, \dots, s_M) \quad .$$

Which, with (3.4), becomes

$$u_r(s_r) = \pi_r \, p_r \, (1 - p_r)^{s_r} \quad . \tag{3.5}$$

If the l^{th} lift is delivered to receiver r, then $v_l = u_r(s_r)$. The initial state of each receiver, $\underline{s}_r$ is given.

The properties of this particular combat value function upon which the result of this prototype model depends is stated in the following assumption:

Assumption 3.4:

Marginal combat values for different receivers are additive, and are non-increasing functions of the weapons state.

5. The Interchange Argument

Now consider a particular ordering O_1 where lift $k + 1$ goes to receiver i, and lift $k + 2$ goes to a different receiver j; and then consider the change in expected value if the recipients of these two lifts are interchanged to give ordering O_2.

From (3.3) the expected value under ordering O_1 can be expressed

$$\begin{aligned} \mathrm{E}\,[V(O_1)] = V_0 &+ v_1 \;\; \mathrm{P}\,[T \geq D_1 + S_1] \\ &+ \dots + v_k \;\; \mathrm{P}\,[T \geq D_k + S_k + W'_{k-1}] \\ &+ u_i(s_i) \;\; \mathrm{P}\,[T \geq \delta_i + \psi_i + W'_k] \\ &+ u_j(s_j) \;\; \mathrm{P}\,[T \geq \delta_j + \psi_j + \delta_i + \rho_i + W'_k] \\ &+ v_{k+3} \;\; \mathrm{P}\,[T \geq D_{k+3} + S_{k+3} + \delta_i + \rho_i + \delta_j + \rho_j + W'_k] \\ &+ \dots + v_L \;\; \mathrm{P}\,[T \geq D_L + S_L + W'_{L-1}] \quad , \end{aligned} \tag{3.6}$$

and the expected value under ordering O_2 is

$$
\begin{aligned}
\mathrm{E}[V(O_2)] = V_0 &+ v_1 \; \mathrm{P}[T \geq D_1 + S_1] \\
&+ \ldots + v_k \; \mathrm{P}[T \geq D_k + S_k + W_{k-1}] \\
&+ u_j(s_j) \; \mathrm{P}[T \geq \delta_j + \psi_j + W_k] \\
&+ u_i(s_i) \; \mathrm{P}[T \geq \delta_i + \psi_i + \delta_j + \rho_j + W_k] \\
&+ v_{k+3} \; \mathrm{P}[T \geq D_{k+3} + S_{k+3} + \delta_i + \rho_i + \delta_j + \rho_j + W_k] \\
&+ \ldots + v_L \; \mathrm{P}[T \geq D_L + S_L + W_{L-1}] \; .
\end{aligned} \tag{3.7}
$$

Subtracting (3.6) from (3.7), the difference in expected value is

$$
\begin{aligned}
\mathrm{E}[V(O_2)] - \mathrm{E}[V(O_1)] &= u_j(s_j) \; \mathrm{P}[T \geq \delta_j + \psi_j + W_k] \\
&+ u_i(s_i) \; \mathrm{P}[T \geq \delta_i + \psi_i + \delta_j + \rho_j + W_k] \\
&- u_i(s_i) \; \mathrm{P}[T \geq \delta_i + \psi_i + W_k] \\
&- u_j(s_j) \; \mathrm{P}[T \geq \delta_j + \psi_j + \delta_i + \rho_i + W_k] \; .
\end{aligned} \tag{3.8}
$$

From (3.8), it is seen that the interchange yields an improvement (i.e., $\mathrm{E}[V(O_2)] > \mathrm{E}[V(O_1)]$) if and only if

$$
\begin{aligned}
&u_j(s_j) \{ \mathrm{P}[T \geq \delta_j + \psi_j + W_k] - \mathrm{P}[T \geq \delta_j + \psi_j + \delta_i + \rho_i + W_k] \} \\
&\quad > \; u_i(s_i) \{ \mathrm{P}[T \geq \delta_i + \psi_i + W_k] - \mathrm{P}[T \geq \delta_i + \psi_i + \delta_j + \rho_j + W_k] \} \; .
\end{aligned} \tag{3.9}
$$

This condition applies for any distribution of T. However, if T has an exponential distribution with mean τ , (3.9) can be reduced into terms separable in i and j. Using $\mathrm{P}[T \geq x] = e^{-x/\tau}$, this condition becomes

$$
\frac{u_j(s_j) \, e^{-(\delta_j + \psi_j)/\tau}}{1 - e^{-(\delta_j + \rho_j)/\tau}} > \frac{u_i(s_i) \, e^{-(\delta_i + \psi_i)/\tau}}{1 - e^{-(\delta_i + \rho_i)/\tau}} \quad . \tag{3.10}
$$

Hence, (3.10) implies the main result of this prototype model (see Ross [Ref. 15, p.18]).

Result 3.1: Under Assumptions 3.1 through 3.4, the sequence of lifts which maximizes the expected combat value is in decreasing order of

$$
\frac{u_r(s_r) \, e^{-(\delta_r + \psi_r)/\tau}}{1 - e^{-(\delta_r + \rho_r)/\tau}} \quad . \tag{3.11}
$$

The ratio of probabilities which multiplies the marginal combat value in (3.11).

$$\frac{e^{-(\delta_r + \psi_r)/\tau}}{1 - e^{-(\delta_r + \rho_r)/\tau}} ,$$

can be thought of as a *logistics weight*, since it is a function of the times taken to transfer the lift (delivery, return and strikedown), and those times are the key logistics factors in the VERTREP problem. Also, to reflect the key ideas of the combat replenishment problem (3.11) will be called a *logistics weighted combat value.*

6. Prototype Model Optimal Sequence

For the prototype combat value function, with marginals given by (3.5), the logistics weighted combat value of a lift is expressed by

$$\frac{\pi_r p_r (1 - p_r)^{s_r} e^{-(\delta_r + \psi_r)/\tau}}{1 - e^{-(\delta_r + \rho_r)/\tau}} ,$$

where the state variable, s_r, takes on n_r consecutive integer values starting with the initial weapons state, $\underline{s}_r$; and n_r, $\underline{s}_r$, π_r, p_r, δ_r, ψ_r, and ρ_r, for $r = 1, \ldots, M$, and τ are all given constants.

To show how the result might be used, an algorithm to obtain the optimal sequence is given in Figure 7.

Step 0 (Initialize):

Input τ, M

$L = 0$ (initialize total number of lifts)

For $r = 1, \ldots, M$ (for each receiver)

Input n_r, $\underline{s}_r$, π_r, p_r, δ_r, ρ_r, ψ_r

$L = L + n_r$ (add up total number of lifts)

$s_r = \underline{s}_r$ (set initial state)

$smax_r = \underline{s}_r + n_r$ (set final state)

$\bar{p}_r = 1 - p_r$ (collect constants)

$$C_r = \frac{\pi_r \ p_r \ e^{-(\delta_r + \psi_r)/\tau}}{1 - e^{-(\delta_r + \rho_r)/\tau}}$$ (collect constants)

$lwcv_r = C_r \ \bar{p}_r^{\ s_r}$ (logistics weighted combat value)

$l = 1$ (initialize: first lift)

Step 1 (For this lift):

$Best = 0$ (initialize best $lwcv$)

For $r = 1, \ldots, M$ (Find optimal receiver)

If $s_r < smax_r$

If $lwcv_r > Best$

$r^* = r$

$Best = lwcv_r$

Print r^* (output: optimal receiver)

$s_{r^*} = s_{r^*} + 1$ (update the state)

$lwcv_{r^*} = C_{r^*} \ \bar{p}_{r^*}^{\ s_{r^*}}$ (update $lwcv$)

Step 2 (Next lift):

If $l < L$

$l = l + 1$

Go to Step 1

Step 3:

Stop

Figure 7. Optimal Sequence Algorithm

Example 3.1. An example of a small Battle Group with only two receivers and few missiles is used to illustrate this model. The given data are shown in Table 1.

Table 1. GIVEN DATA FOR EXAMPLE 3.1

Receiver (r)	1	2
Initial State (s_r)	0	2
Requests (n_r)	4	4
Delivery Time (δ_r)	.30	.15
Return Time (ρ_r)	.30	.15
Strikedown Time (ψ_r)	.25	.10
Single-shot kill Probability (p_r)	.65	.40
Engagement Probability (π_r)	.33	.67
Expected time between raids = 1.0		

The optimal sequence algorithm of Figure 7 is used to find the ordering of lifts which maximizes the expected combat value of weapons available in the battle group when combat commences. The initial weapons state of the Battle Group is

$$(s_1, s_2) = (0, 2) \quad .$$

The marginal combat value of a lift to receiver 1 is

$$\begin{aligned} u_1(s_1) &= \pi_1 \, p_1 \, (1 - p_1)^{s_1} \\ &= (.33)(.65)(1 - .65)^0 \\ &= .214 \quad . \end{aligned}$$

The marginal combat value of a lift to receiver 2 is

$$\begin{aligned} u_2(s_2) &= \pi_2 \, p_2 \, (1 - p_2)^{s_2} \\ &= (.67)(.4)(1 - .4)^2 \\ &= .096 \quad . \end{aligned}$$

The logistics weighting factor for a lift to receiver 1 is

$$\frac{e^{-(\delta_1 + \psi_1)/\tau}}{1 - e^{-(\delta_1 + \rho_1)/\tau}} = \frac{e^{-(.30+.25)/(1.0)}}{1 - e^{-(.30+.30)/(1.0)}} = 1.28 \quad ;$$

and the logistics weighted combat value is (.214)(1.28) = .274 .

The logistics weighting factor for a lift to receiver 2 is

$$\frac{e^{-(\delta_2 + \psi_2)/\tau}}{1 - e^{-(\delta_2 + \rho_2)/\tau}} = \frac{e^{-(.15+.10)/(1.0)}}{1 - e^{-(.15+.15)/(1.0)}} = 3.00 \quad ;$$

and the logistics weighted combat value is (.096)(3.00) = .288 .

Thus, since .288 > .274 , it is optimal for the first lift to be dispatched to receiver 2. The state of receiver 2 is then incremented by one, and the second lift is considered; and so forth. A Fortran implementation of the optimal sequence algorithm is provided in Appendix A. The output of the program is the optimal ordering, O_* , which tells the decision maker to dispatch lifts to receivers 1 and 2 in the following sequence:

$$O_* = \{2,\ 1,\ 2,\ 2,\ 1,\ 2,\ 1,\ 1\}$$

Table 2 on page 26 shows the numerical results of the replenishment process under the optimal ordering.

Table 2. RESULTS FOR EXAMPLE 3.1

Lift number (l)	0	1	2	3	4	5	6	7	8
Receiver (r)	-	2	1	2	2	1	2	1	1
Dispatch Time (W_{l-1})	-	0.00	0.30	0.90	1.20	1.50	2.10	2.40	3.00
Completion Time ($CT_l = W_{l-1} + \delta_r + \psi_r$)	0.00	0.25	0.85	1.15	1.45	2.05	2.35	2.95	3.55
Marginal value ($v_l = u_r(s_r)$)	-	.096	.214	.058	.035	.075	.021	.026	.009
Probability of completion $P[T \geq CT_l]$	1.00	.779	.427	.317	.235	.129	.095	.052	.029
State after completion (s_1, s_2)	(0,2)	(0,3)	(1,3)	(1,4)	(1,5)	(2,5)	(2,6)	(3,6)	(4,6)
Total Combat Value $V_l = U(s_1, s_2)$	.429	.525	.740	.798	.832	.907	.928	.955	.964
Probability of raid arrival in this state $P[CT_l \leq T < CT_{l+1}]$	.221	.351	.111	.082	.106	.033	.043	.024	.029
Expected Combat Value = .635									

7. Interpretation and the Exponential Assumption

The assumption of the exponential distribution of T, which possesses the memoryless property, was necessary to get the form of (3.10) in which the terms are separable in i and j. Although the result, (3.10), was obtained directly from (3.9), using the stated assumptions, a derivation of an intermediate form is useful to show clearly the necessity of the memoryless property of the exponential distribution of T, and offer some interpretations of the results.

Using the definition of conditional probability, the first probability term in (3.9) can be expanded as follows:

$$P[T \geq \delta_j + \psi_j + W_k] = P[T \geq \delta_j + \psi_j + W_k \mid T \geq W_k] \times P[T \geq W_k] \quad . \qquad (3.12)$$

The last probability term in (3.9) can be expanded as follows:

$$\begin{aligned} P[T \geq \delta_i + \psi_i + \delta_j + \rho_j + W_k] \\ &= P[T \geq \delta_i + \psi_i + \delta_j + \rho_j + W_k \mid T \geq \delta_j + \rho_j + W_k] \\ &\times\ P[T \geq \delta_j + \rho_j + W_k \mid T \geq W_k] \times P[T \geq W_k] \ . \end{aligned} \tag{3.13}$$

Similarly, for the other probability terms in (3.9).

Interpretations may now be made. Recalling that W_k is defined as the partial sum of delivery and return times through the k^{th} lift, it may also be interpreted as the *dispatch* time for the $k+1^{st}$ lift. Then the probability, $P[T \geq W_k]$, may be interpreted as the probability that a $k+1^{st}$ lift can be dispatched.

The conditional probability term in (3.12), $P[T \geq \delta_j + \psi_j + W_k \mid T \geq W_k]$ may then be interpreted as the probability that a transfer to receiver j can be completed, if it gets dispatched at time W_k.

Similarly, the first conditional probability term in (3.13), $P[T \geq \delta_i + \psi_i + \delta_j + \rho_j + W_k \mid T \geq \delta_j + \rho_j + W_k]$ may then be interpreted as the probability that a transfer to receiver i can be completed, if it gets dispatched when the helo returns from its round-trip to receiver j.

In a similar manner, the second conditional probability term in (3.13) $P[T \geq \delta_j + \rho_j + W_k \mid T \geq W_k]$ may be interpreted as the probability that a helicopter round-trip to receiver j can be completed, if it gets dispatched at time W_k.

This representation also implies that the decision concerning the choice between ordering O_1 and O_2 may be interpreted as deciding, at time W_k, which receiver gets the next lift to be dispatched.

Getting back to (3.9), if all probability terms are expanded, it can be seen that the factor $P[T \geq W_k]$ is common to every term and therefore may be divided out. Then (3.9) becomes

$$\begin{aligned} &u_j(s_j)\left[\begin{array}{l} P[T \geq \delta_j + \psi_j + W_k \mid T \geq W_k] \\ \quad - \left[\begin{array}{l} P[T \geq \delta_j + \psi_j + \delta_i + \rho_i + W_k \mid T \geq \delta_i + \rho_i + W_k] \\ \qquad \times\ P[T \geq \delta_i + \rho_i + W_k \mid T \geq W_k] \end{array}\right] \end{array}\right] \\ &> u_i(s_i)\left[\begin{array}{l} P[T \geq \delta_i + \psi_i + W_k \mid T \geq W_k] \\ \quad - \left[\begin{array}{l} P[T \geq \delta_i + \psi_i + \delta_j + \rho_j + W_k \mid T \geq \delta_j + \rho_j + W_k] \\ \qquad \times\ P[T \geq \delta_j + \rho_j + W_k \mid T \geq W_k] \end{array}\right] \end{array}\right] . \end{aligned} \tag{3.14}$$

It is here that the assumption that T has an exponential distribution is invoked. The memoryless property of the exponential implies that the second conditional probability terms on each side of (3.14) can be reduced as follows (further reductions are possible, but this intermediate step is taken to derive a form of the logistics weights with a useful interpretation):

$$P[T \geq \delta_j + \psi_j + \delta_i + \rho_i + W_k \mid T \geq \delta_i + \rho_i + W_k] = P[T \geq \delta_j + \psi_j + W_k \mid T \geq W_k] \quad (3.15)$$

$$P[T \geq \delta_i + \psi_i + \delta_j + \rho_j + W_k \mid T \geq \delta_j + \rho_j + W_k] = P[T \geq \delta_i + \psi_i + W_k \mid T \geq W_k] \quad (3.16)$$

Using these reductions, (3.14) simplifies to

$$\begin{aligned} & u_j(s_j)\left[\begin{array}{l} P[T \geq \delta_j + \psi_j + W_k \mid T \geq W_k] \\ \quad - \left[\begin{array}{r} P[T \geq \delta_j + \psi_j + W_k \mid T \geq W_k] \\ \times\ P[T \geq \delta_i + \rho_i + W_k \mid T \geq W_k] \end{array}\right] \end{array}\right] \\ & > u_i(s_i)\left[\begin{array}{l} P[T \geq \delta_i + \psi_i + W_k \mid T \geq W_k] \\ \quad - \left[\begin{array}{r} P[T \geq \delta_i + \psi_i + W_k \mid T \geq W_k] \\ \times\ P[T \geq \delta_j + \rho_j + W_k \mid T \geq W_k] \end{array}\right] \end{array}\right] . \end{aligned} \quad (3.17)$$

Then, as a direct consequence of the memorylessness of T, the terms, $P[T \geq \delta_i + \psi_i + W_k \mid T \geq W_k]$, and $P[T \geq \delta_j + \psi_j + W_k \mid T \geq W_k]$ can be factored out, separating terms in i and j.[2] Equation (3.17) can then be rearranged as

$$\frac{u_j(s_j)\, P[T \geq \delta_j + \psi_j + W_k \mid T \geq W_k]}{1 - P[T \geq \delta_j + \rho_j + W_k \mid T \geq W_k]} > \frac{u_i(s_i)\, P[T \geq \delta_i + \psi_i + W_k \mid T \geq W_k]}{1 - P[T \geq \delta_i + \rho_i + W_k \mid T \geq W_k]} \ . \quad (3.18)$$

Thus the logistics weighted combat value of the alternative lifts evaluated for dispatch at time W_k is

$$\frac{u_r(s_r)\, P[T \geq \delta_r + \psi_r + W_k \mid T \geq W_k]}{1 - P[T \geq \delta_r + \rho_r + W_k \mid T \geq W_k]} \ , \quad (3.19)$$

for receivers $r = 1, \ldots, M$.

---

2 The separation also relies on factoring out the marginal utilities, and assumptions about the delivery and return times, which are discussed later.

The conditional probabilities in (3.19) provide an interpretation which is reasonable for determining the priority of a lift to be dispatched at time W_k. The probability in the numerator may be read as the conditional probability that a lift dispatched at time W_k can have strikedown completed before a raid arrival, given that it can be dispatched before a raid arrival. This will be referred to simply as the conditional probability of strikedown completion. The probability in the denominator may be read as the conditional probability that a helicopter dispatched at time W_k can complete that transfer and be ready for another lift before a raid arrival, given that it can be dispatched before a raid arrival. This will be referred to simply as the conditional probability of round-trip completion.

The logistics weight which is applied to combat value is thus

$$\frac{\text{(conditional probability of strikedown completion)}}{1 - \text{(conditional probability of round-trip completion)}} \ .$$

This is consistent with the intuitive idea that a lift with a shorter strikedown time should have a higher logistics weight than a lift with a longer strikedown time, and a lift that consumes less helicopter round-trip time (which allows subsequent lifts to be transferred sooner) should have a higher logistics weight than a lift that consumes more helicopter round-trip time.

8. Transfer Time Assumptions

Referring back to (3.6) and (3.7), it can be seen that the assumption that delivery and return times be constant was necessary so that the sum $\delta_i + \rho_i + \delta_j + \rho_j + W_k$ which enters the terms for the $k+3$ rd and subsequent lifts, does not depend on the order of deliveries to receivers i and j. That is, the δ_j and ρ_j, for example, can not depend on when that delivery starts. This permits those later terms to cancel when the difference in expected value is taken.

The assumption that strikedown times (as well as the delivery times) be constant was necessary so that the ψ_j, for example, that appears in the conditional probability terms

$$P\,[T \geq \delta_j + \psi_j + W_k \,|\, T \geq W_k] \ ,$$

and

$$P[T \geq \delta_j + \psi_j + \delta_i + \rho_l + W'_k \mid T \geq \delta_l + \rho_l + W'_k] \quad ,$$

in (3.14), do not depend on the time at which the strikedown starts. Consequently, both of these conditional probabilities equal $P[T \geq \delta_j + \psi_j]$, which in turn permits separating terms in i and j. This is the motivation for the assumption that strikedown queues not develop on receivers. If strikedown queues could develop, then earlier lifts might delay the strikedown of subsequent lifts. Then strikedown times would depend on when and how many previous lifts were delivered. Strikedown queues could be precluded by *blocking* subsequent deliveries to a receiver if a strikedown is not completed, but it is also assumed that there be no blocking. This is necessary because if an earlier lift precluded a subsequent delivery to any particular receiver, then some orderings would be disallowed, and some interchanges may be blocked. Thus the interchange argument could not be applied.

The condition $S_l \leq R_l$, for all l , which further implied the condition $\psi_r \leq \rho_r$, for all r , is sufficient to ensure that no strikedown queueing (or blocking) could occur, but is stronger than it has to be. What is needed to preclude blocking and strikedown queues is to require that a lift to any particular receiver have strikedown completed before another lift can be delivered to the same receiver. Referring to Figure 5 on page 16 it is seen that the necessary condition for there to be no blocking or strikedown queueing is, for consecutive lifts to the same receiver, that

$$S_l \leq R_l + D_{l+1} \quad , \text{for } l = 1, \dots , L-1 \quad ;$$

which in terms of a specific receiver becomes

$$\psi_r \leq \rho_r + \delta_r \quad , \text{for } r = 1, \dots , M \quad . \tag{3.20}$$

For this model, it is also necessary that strikedown completions occur in the same order in which deliveries are dispatched (i.e., an ordering O_* refers both to the order in which transfers are started and completed). This requirement permits the completion times shown in Figure 6 on page 17 to be ordered, the combat value to be expressed as in (3.1) and (3.2), and the interchange argument to be applied. As with strikedown queueing and blocking, the condition $S_l \leq R_l$, for all l , which further implied the condition $\psi_r \leq \rho_r$, for all r , is sufficient to ensure that the ordering is maintained, but again stronger than necessary. Referring to Figure 5 on page 16 it is easily seen that the necessary condition for this ordering to be maintained is that

$$S_l \leq R_l + D_{l+1} + S_{l+1} \quad , \text{for } l = 1, \dots, L-1 \quad . \tag{3.21}$$

To see what this condition implies in terms of lifts to specific receivers, two cases ca be considered.

Case I: If consecutive lifts go to distinct receivers in the order i followed by j, (3.21) becomes

$$\psi_i - \rho_i \leq \delta_j + \psi_j \quad , \text{for } i = 1, \dots, M \quad j = 1, \dots, M \quad i \neq j \quad . \tag{3.22}$$

Case II: If consecutive lifts go to the same receiver, then the ordering will be maintained due to the condition given by (3.20) which precluded strikedown queueing or blocking.

Thus the following relaxation of Assumption 3.2 is applicable for Result 3.1:

Assumption 3.2′: To preclude the development of strikedown queues, and maintain the same ordering from the start of a transfer to its completion,

$$\psi_r \leq \rho_r + \delta_r \quad , \text{for } r = 1, \dots, M \quad ,$$

and

$$\psi_i - \rho_i \leq \delta_j + \psi_j \quad , \text{for } i = 1, \dots, M \quad j = 1, \dots, M \quad i \neq j \quad .$$

9. Combat Value Function Assumptions

Although a very specific combat model was used to derive a simple Battle Group combat value function for this model, only two properties of the function were necessary to the derivation of Result 3.1. The first property was additivity. It was necessary that the marginal combat values of lifts for each receiver not depend on the state of other receivers. The implication of this is that the total combat value for the battle group is the sum of the combat values of the receivers (i.e., there are no *cross terms* in

the Battle Group combat value function). In this model, the additivity property was due to the assumption that attacker engageability by the defenders was mutually exclusive. The property was implicitly used in the interchange argument in (3.6) and (3.7), where the marginal value of the lift for, say, receiver i was the same whether that lift came before or after the lift for receiver j. Consequently, the marginal values factored out in (3.9), and ultimately permitted separating terms in i and j.

The other necessary property was the concavity of the Battle Group combat value function. This property is necessary to preclude scheduling a receiver to get, say, his second replenishment missile before his first one.

The assumption that marginal values are non-negative (and hence that the combat value function is non-decreasing) was used in this model, but is easily relaxed. Since the sequence that maximizes expected combat value is in decreasing order of the logistics weighted combat values, and the weights (a ratio of probabilities) are always non-negative, the replenishment process can simply be terminated after the last non-negative valued lift.

Defining the combat value function as a probability of successful defense was completely arbitrary. Any utility function satisfying the additivity and concavity assumptions could have been used for the Battle Group combat value function. Appendix B presents a heuristic method that can be used to derive combat values, but that does not require a fine-grained specification of the Battle Group defense formation and specific raid parameters.

B. A VERTREP SCHEDULING HEURISTIC

Some of the conditions under which the prototype model gives an optimal sequence are too restrictive for a real problem. However, even if all the conditions do not hold exactly, they may be close, and sequencing the lifts in decreasing order of logistics weighted combat value, hereafter LWCV, may give good, if not optimal, results.

In this section, a more general VERTREP problem is defined, and a heuristic scheduling algorithm which uses the LWCV criteria is outlined. The algorithm has been implemented, and a Battle Group example is presented. To examine how good a schedule the LWCV heuristic produces, a method taken from combinatorial optimization, called a *local neighborhood search*, is used to improve the solution for the example,

and the results compared. The chapter is then concluded with a discussion of the conditions under which the LWCV heuristic may be expected to produce a good schedule.

1. The Battle Group Combat VERTREP Problem

A typical Battle Group consists of several combatant ships (receivers) being supported by one multi-product combat logistics ship (deliverer) with a few logistics helicopters. Each combatant ship has several anti-aircraft and anti-missile weapons systems installed to provide a layered defense against air attack, as well as weapons for use against hostile submarines and surface units.

Following an air attack, each receiver requests multiple lifts of each type of ammunition used. Aboard the delivery ship, different types of ammunition may take different amounts of time to breakout and prepare for transfer. Aboard the receiving ships, each type of ammunition is processed separately, and may develop its own strikedown queue.

When the air attack ends, which marks the start of a replenishment period, each combatant ship is in the vicinity of an assigned combat station which may be at a great distance from the logistics ship. At that time, the Battle Group Commander issues maneuvering orders based perhaps on tactical considerations, which determines the relative positions and relative speeds of the delivery ship and receiving ships for the duration of the replenishment period. For example, the delivery ship may be ordered to proceed on a particular course and speed, and the receivers may be ordered to close the delivery ship as fast as possible for replenishment, remain close for a while, and then proceed to take up new defensive positions by a particular time.

The time available to conduct ammunition transfers, the air raid interarrival time, is a random variable, with an arbitrary distribution. When another air raid arrives, it terminates the replenishment process; transfers in progress are not completed.

From a scheduling theory viewpoint, job processing times are sequence-dependent. That is, several components of the total time it takes to process each VERTREP lift, depend on the lifts that have been sequenced ahead of them. Specifically, helicopter transfer time includes variable flight time which depends on the range of the receiver at the time of delivery. The range, in turn, depends on the time that the lift is dispatched, which depends on the time consumed by previous lifts. Also, the time that a receiver takes from when a lift is dropped off, until strikedown is completed depends on how long that lift must remain in a strikedown queue, which depends on when and how many previous lifts were delivered.

This feature of sequence-dependent processing times in a job sequencing problem implies that the general Battle Group combat VERTREP problem is equivalent to the *traveling salesman problem*; see Conway, Maxwell, and Miller [Ref. 7, pp. 53-66]. In the combat setting of this problem, there may be as many as a few hundred lifts to schedule, and considering that with another attack anticipated, replenishment should commence as soon as the previous raid ends, a solution to this scheduling problem is needed quickly. Hence, the heuristic approach.

2. The Combat VERTREP Scheduling Heuristic

This heuristic for scheduling combat VERTREP is conceptually the same as the sequencing algorithm given in Figure 7 for the prototype model.

a. Inputs. The following inputs are required:

(1) An attack interarrival time distribution and estimates of the parameters of the distribution.

It is assumed that the attack interarrival time is a random variable with range over the positive real line; for example, the exponential distribution or the gamma distribution.

(2) An arbitrary Battle Group combat value function.

No assumptions are made concerning the form of the combat value function. However, a sensible combat value function would be a non-decreasing concave function of the weapons state (up to the weapons capacity of each receiver), and an increasing function of weapon system and defender effectiveness. Appendix B provides a discussion of the concepts and characteristics of a combat value function, and suggests a heuristic method of calculating combat values for every possible lift of ammunition in the Battle Group. Other forms of a combat value function may also be used. Since this scheduling heuristic uses a forward induction procedure, an arbitrary combat value function which depends on the current weapons state of the entire Battle Group could be used.

(3) Battle Group maneuvering orders.

It is assumed that the initial positions of each receiver relative to the delivery ship are known at the outset, and that the maneuvering of the delivery ship and

receiving ships during the replenishment period has been specified. In addition to that input, to determine the variable helicopter flight time from delivery ship to each receiver and back at any time, the algorithm needs each receiver's relative closing and opening speeds, and relative helicopter speeds, out from the delivery ship with a lift, and returning to the delivery ship empty. The relative speeds may be computed given the following data: delivery ship true course and speed, receiving ships true stationing speed, helicopter true air speed (unloaded and loaded, with possibly different speeds for different types of lifts), and true wind speed and direction; see Defense Mapping Agency Pub. 217 [Ref. 22].

(4) Ammunition requirements and handling times.

It is assumed that the ammunition requests and fixed handling times for all lifts are given before the generation of an initial schedule or a revision. The first input is the number of receivers. Then, for each receiver, the number of types of weapons must be given. Then, for each type of weapon on each receiver, a time to strikedown must be given, as well as that receiver's capacity and current weapons state (from which the number requested is determined). Inputs related to delivery include the number of helicopters, and for each type of ammunition carried by the delivery ship, the delivery ship breakout time, and fixed handling times for a helicopter to pick up a lift, and drop off a lift. Note that the total amount of time that a helicopter takes with a VERTREP lift includes fixed time to pick up plus variable flight time out to the receiving ship, and fixed time to drop off plus variable flight time back to the delivery ship. The next section discusses the computation of total transfer times from the given inputs. In addition to the basic receiving and delivery inputs, initial conditions, which include specifying breakout status, helicopter status, and strikedown queue status for each weapon on each receiver, can be used to revise a schedule once a VERTREP is in progress.

b. Transfer Times. At each helicopter dispatch time, the LWCV is computed for the next requested lift of every requested ammunition type for every receiver. To do this, besides the combat value of each lift, the helicopter round-trip completion time, and strikedown completion time are needed. In the prototype model, delivery ship breakout times were disregarded, strikedown queues were precluded, and the receivers remained on fixed stations, so that all the times were fixed constants. In this more general problem, transfer times must be computed. The approach used is to represent the VERTREP process, from breakout at the delivery ship to strikedown at the receiving ship, as a

deterministic network, or PERT type system; see, for example, Elmaghraby [Ref. 23]. The desired transfer times which, in project scheduling terminology are events, are then obtained by a partial forward pass on the network. A helicopter may be dispatched at the later of its return from a previous lift or breakout completion of the current lift, plus the fixed time it takes the helicopter to pick up the lift. The time at which a lift is dropped off at the receiver is the sum of the time of dispatch plus variable flight time to the receiver plus the fixed time to drop off the lift. The event which marks the time of strikedown completion is the length of the current lift strikedown activity time added to the later of the previous strikedown completion event or the current lift drop off time. And finally, the event that marks the helicopter's return from the current round-trip and readiness to pick up the next lift is the variable flight time returning added to the event time when the current lift was dropped off. Details are provided in Appendix C.

c. Logistics Weights. The key element of this scheduling heuristic is the use of LWCV as a dynamic allocation index. Having already discussed the generation of combat values, it remains to consider the form of the *logistics weights.* From the prototype model, the form of the logistics weights in (3.19) is

$$\frac{\mathrm{P}\,[T \geq \delta_r + \psi_r + W_k \,|\, T \geq W_k]}{1 - \mathrm{P}\,[T \geq \delta_r + \rho_r + W_k \,|\, T \geq W_k]} \quad .$$

Although this form of the logistics weight was derived for exponential air raid interarrival times, and is only exact in that case, it will be used as a heuristic for general distributions defined on the positive real line as well. Conditions under which this heuristic may be expected to produce a good schedule are discussed in the final section of this chapter.

The conditional probabilities are derived from the unconditional probabilities as follows:

$$\mathrm{P}\,[T \geq \delta_j + \psi_j + W_k \,|\, T \geq W_k] = \frac{\mathrm{P}\,[T \geq \delta_j + \psi_j + W_k]}{\mathrm{P}\,[T \geq W_k]} \quad ,$$

and

$$\mathrm{P}\,[T \geq \delta_j + \rho_j + W_k \,|\, T \geq W_k] = \frac{\mathrm{P}\,[T \geq \delta_j + \rho_j + W_k]}{\mathrm{P}\,[T \geq W_k]} \quad .$$

d. The Algorithm. The algorithm for the combat VERTREP scheduling heuristic is outlined as follows

(1) Read inputs and initialize.
(2) For the next available helicopter:
  - For each receiver:
    - For each weapon type:
      - Obtain the combat value of the next lift requested.
      - Calculate receiver motion and transfer times.
      - Calculate logistics weights.
      - Calculate LWCV.
  - Schedule the lift with maximum LWCV.
  - For the scheduled receiver / weapon:
    - Increment the weapon state.
    - Set the time of breakout completion.
    - Set the time of strikedown completion.
  - Set the time of helicopter return.

(3) If there are more lifts requested, go to (2).
(4) Write the schedule.
(5) Stop.

3. A Battle Group Example

This example considers the scheduling of VERTREP for a Battle Group consisting of four receiving ships each of which has four types of weapons to be replenished (from a computational standpoint, this is equivalent to eight receivers each with two types of weapons, or sixteen receivers each with one type of weapon, etc.). There is one delivery ship, one helicopter, and the total number of lifts requested is 97. The air raid interarrival time is assumed to be exponentially distributed with an expected air raid arrival of 4 hours. In this example, the four receivers are called *Ship1, Ship2, Ship3,* and *Ship4*, and the seven different types of ammunition within the Battle Group, are called *WepA, WepB, WepC, WepD, WepE, WepF,* and *WepG*. Appendix D contains the inputs and resulting schedule for this example.

The first seven tables in Appendix D represent the inputs and computation of marginal combat values using the priority list method of Appendix B. The Battle Group ammunition summary is shown in Table 6 on page 191. The prioritized list of ammu-

nition by serial number for each of the receivers is shown in Table 7 on page 192 through Table 10 on page 196. The combined list, sorted by receiver priority, with Battle Group priorities assigned, is shown in Table 11 on page 197. The combined list, sorted by Battle Group priority, with marginal combat values calculated is shown in Table 12 on page 200.

The next three tables give the logistics inputs. The ammunition requests from each receiver for each weapon type, and the time to strikedown each type of weapon are shown in Table 13 on page 203. For each type of weapon, the time for the delivery ship to break out a lift, and the fixed times for a helicopter to pick up and drop off a lift are shown in Table 14 on page 203. In lieu of delivery ship course and speed, receiving ship stationing speed and initial and final station range and bearing, helicopter true air speed, and true wind speed and direction, Table 15 on page 204 summarizes the relative speeds and ranges determined from the Battle Group maneuvering orders.

The initial VERTREP schedule obtained with the LWCV heuristic is shown in Table 16 on page 205. The example was run on a Compaq Portable II computer with an 80286 CPU running at 12 Mhz. and an 80287 math co-processor. The schedule was produced in 2.52 seconds.

The nature of the schedule reflects all of the considerations that have entered the modeling of this problem. For example, the first lift of WepA to Ship1, the weapon with the highest combat potential in the Battle Group, occurs on the sixteenth helicopter delivery, which is in contrast to the first four consecutive scheduled lifts being weapons with much lower combat potential which are delivered to Ship4. The explanation for this involves the relative weapons states of all the receivers, and the logistics transfer times. With respect to weapons state, seen in Table 6 on page 191, Ship1 starts off initially with ten of WepA, so that the first one requested would be the eleventh. In contrast to this, each of the initial lifts to Ship4 start off in a weapon state of zero. This distinction is reflected in the marginal combat values in Table 12 on page 200. With respect to logistics transfer times, as seen in Table 15 on page 204, Ship1 starts off initially at a much greater range from the delivery ship than any of the other receivers, so that an earlier lift to Ship1 would consume much more helicopter flight time than lifts to the other receivers. Consequently, the lifts for Ship1 get a lower logistics weight. The entire schedule may be similarly analyzed.

4. Improvement by Local Neighborhood Search

Whereas it appears, intuitively, that the schedule produced by the LWCV heuristic is not bad, it remains to be judged quantitatively how good it is. To get an approximate idea how close the initial schedule is to optimality, a method taken from combinatorial optimization, called a local neighborhood search, is used; see, for example, Kohler and Steiglitz [Ref. 24] or Parker and Rardin [Ref. 25].

The general strategy of a local neighborhood search in a scheduling problem is to start with some initial schedule, search in some chosen neighborhood of that schedule, adopting improvements as they are found, and continuing until no further local improvements are possible in that neighborhood. For example, the smallest neighborhood for a scheduling problem is the set of schedules obtained by interchanging two adjacent jobs.

The variant of local neighborhood search used here is based on the classic *k-opt* algorithm of Lin [Ref. 26] for the traveling salesman problem. Following Lin, a schedule is called k-opt if it is impossible to obtain a schedule which improves the value of the objective function by interchanging any k of the jobs. In the following, the initial schedule obtained with the LWCV heuristic is compared to the 2-opt and 3-opt schedules. (Lin's results say that 4-opt schedules are not worth generating, in that they require much more time to produce, and that their probability of being optimal is not noticeably better than for the 3-opt schedule.)

The performance of the initial schedule compared to the local neighborhood search improvement is shown in Table 3.

Table 3. INITIAL SCHEDULE VS. LNS IMPROVEMENT

Schedule	Initial LWCV Heuristic	2-Opt LNS Improvement	3-Opt LNS Improvement
Run time	~ 3 sec.	~ 39 min.	> 23 hr.
Expected Combat Value	17336.5	17871.8	17882.3
Percent 3-Opt	96.9%	99.9%	100%

The 2-opt VERTREP schedule is shown in Table 17 on page 207 and the 3-opt schedule is shown in Table 18 on page 209. They may be directly examined to see how they differ from the initial schedule listed in Table 16 on page 205, however a snapshot of projected completions at a particular time provides a good picture of the qualitative differences between the initial and *k*-opt schedules. The projected number of strikedown completions for each weapon type on each receiver at the expected time of the next raid arrival is summarized in Table 4. The numbers in parentheses show additional lifts for which the transfer is projected to be completed by E(T), but for which strikedown is not.

Table 4. TRANSFERS COMPLETED AT E(T): LWCV HEUR. VS K-OPT

Receiver	Ammo Type	Strikedown Completions Heur.	2-Opt	3-Opt
Ship1	WepA	4 (+2)	6 (+1)	6 (+1)
Ship1	WepD	2	2	2
Ship1	WepE	2	2	2
Ship1	WepF	0	0	0
Ship2	WepB	5 (+1)	5	5
Ship2	WepD	0	0	0
Ship2	WepE	0	0	0
Ship2	WepF	0	0	0
Ship3	WepC	5	5	5
Ship3	WepD	2	2	2
Ship3	WepE	2	2	2
Ship3	WepF	0	0	0
Ship4	WepC	4	4	4
Ship4	WepD	1	1	1
Ship4	WepE	2	2	2
Ship4	WepG	0	0	0

This summary shows the qualitative improvement achieved by the local neighborhood search. At this particular time, the improved schedules permit two more strikedown completions of WepA on Ship1, and avoid the development of a strikedown queue. This is a noteworthy improvement since that is the weapon with the highest combat potential and longest strikedown time in this example. However, looking over

the entire table, it may be seen that the initial schedule is reasonably close to the 2-opt and 3-opt schedules which took so much longer to generate. This summary does not show any difference between the transfer completions under the 2-opt and 3-opt schedules, because, as may be seen in Table 17 on page 207 and Table 18 on page 209, they differ only by a permutation of a few lifts which slightly improves the objective function value, but does not change the number of projected completions prior to the expected time of raid arrival.

It should be noted that the exhaustive all-pairs and all-triples interchange searches used here are certainly not the only alternative to staying with the initial schedule. A great many possibilities that exploit the characteristics of the combat VERTREP problem can be easily envisioned to heuristically improve the initial schedule within a user specified reasonable time. The purpose here was to get a feeling for how good an initial schedule the LWCV heuristic generated. Heuristic improvement of the initial schedule is left for future consideration.

5. Favorable LWCV Conditions

In this section, the conditions under which the LWCV heuristic may be expected to produce a good schedule are discussed. These conditions are based on the prototype model assumptions which permitted the use of the interchange argument in sequencing and ultimately the separation of terms leading to the LWCV optimality criteria.

a. Combat Value Function. With respect to the Battle Group combat value function, the LWCV criteria is exact if the marginals are additive, as in the prototype combat model, or the combat value priority list method of Appendix B. For an arbitrary combat value function which may not have additive marginals, the LWCV criteria may still be good if cross terms due to small increments in the weapons state of other receivers are not appreciable.

b. Air raid interarrival time distribution. With respect to the air raid interarrival time distribution, the LWCV criteria is exact with the exponential distribution. Markovian interarrival times may be a plausible assumption based on the following conditions:

Next missile attack may be submarine launched and occur at any time.

Next attack may be by stragglers from the last wave.

Next wave may be another bomber regiment with uncertain interarrival time.

Next wave may be the same bomber regiment after an uncertain turnaround.

Next wave may be delayed indefinitely (i.e., an "independent" subsequent attack not directly tied to the previous wave.)

For an arbitrary distribution, the LWCV criteria may still be good if the conditional probabilities statements in Equations (3.15) and (3.16) are approximately true. This might be the case if helicopter round-trip time were insignificant with respect to the other terms in (3.15) and (3.16), which could occur when the receivers have closed the delivery ship -- the situation in which VERTREP is the most efficient.

c. Strikedown Queues. With respect to the development of strikedown queues for each weapon on each receiver, the LWCV criteria is exact if no strikedown queue exists whenever a receiver has a lift dropped off. Strikedown queues are precluded (trivially) for each receiver's first delivery. Strikedown queues are precluded if helicopter round-trip time is longer than weapon strikedown time (as was assumed in the prototype model). Strikedown queues may be avoided even if helicopter delivery cycle is shorter than receiver strikedown time, if the resulting sequence spreads out deliveries within the Battle Group such that intervening deliveries of other weapons to the same receiver and deliveries to other receivers delay subsequent deliveries of the weapon with the long strikedown time. This might tend to occur due to a combat value function which tries to balance the weapons states in the Battle Group.

d. LWCV Numerator. With respect to the product of marginal combat value and conditional probability of strikedown completion in the LWCV numerator, the LWCV criteria is exact under the conditions discussed above for each term individually, and may be good if both individual terms are good, also as discussed above. In addition to those conditions, the LWCV criteria may be good anyway if the departures from exactness of the two terms offset each other in the product. This may occur if marginal combat values are decreasing (as will usually be the case in the setting of this problem) and since any strikedown queue will diminish with time, the conditional probability of completing strikedown of the next lift may be increasing in short time periods around each dispatch time. Hence, the product may give a good approximation even if both terms are not individually good enough.

e. Later Lifts. The interchange argument required the expected value contribution of later lifts before and after the interchange to cancel out. This condition is exact if the sum of helicopter round-trip times to any two receivers, in either order, are equal, and if no strikedown queues develop for later lifts. The conditions related to strikedown queues are the same as discussed above. The equality of the sum of helicopter round-trip times is guaranteed if the receivers are non-maneuvering during the entire replenishment process. It may also be the case during portions of the replenishment process when the receivers are steady on replenishment stations, or combat stations. Even when the receivers are maneuvering, the sum of round-trip times may be approximately equal if the relative motion of receivers i and j with respect to the delivery ship are similar (i.e., both opening or both closing at close relative speeds).

Collectively, the conditions under which the LWCV heuristic may be expected to produce a good schedule cover a great many possibilities that may be encountered in a real problem.

IV. THE COMBAT CONREP PROBLEM: STOCHASTIC SCHEDULING OF GROUPED JOBS

A model is developed in which the ships of the Battle Group are scheduled for connected replenishment (CONREP). Dynamic programming is used to maximize the expected combat value of the weapons available at the stochastic time when the replenishment terminates and combat commences. In the terminology of flow shop scheduling, the time available to conduct the replenishment determines a stochastic *due date*, and the objective, or scheduling criterion, is equivalent to *minimizing the weighted number of late jobs*; the weights in this model are called *marginal combat values*. The decisions include the optimal partitioning of the receivers into sets assigned to each available delivery ship side, the optimal order in which receivers are sequenced alongside the delivery ship within each set, and the optimal allocation of time alongside the delivery ship to each receiver. These decisions collectively will be referred to as the *schedule*. Initially, a schedule is developed which specifies the optimal decisions for the entire process given the information available at the outset. This type of schedule is what Pinedo [Ref. 10] calls a *static list policy* since the schedule can be thought of as arranging all the jobs to be performed in a list in the order in which they will be performed from the start. Since additional information may become available as the process proceeds, the model will ultimately be extended to include *dynamic* revision of the schedule.

A. DESCRIPTION OF THE COMBAT CONREP PROBLEM

The problem is scheduling CONREP within a Battle Group which contains one delivery ship and several receiving ships. The delivery ship can conduct two connected replenishments in parallel (literally) at port and starboard side replenishment stations. Each receiver requests several deliveries, or *lifts*, of ammunition. The time available to conduct ammunition transfers, the air raid interarrival time, is a random variable. The times to conduct transfers to each receiver, which include delivery times and receiver strikedown times are assumed to be deterministic. When an air raid arrives it terminates the replenishment process; transfers in progress are not completed. While alongside the delivery ship, each receiver gets some consecutive number of lifts, which may be thought of as *grouped jobs*.

B. OPTIMAL INITIAL SCHEDULE

A model to determine an optimal initial static list schedule is built up in steps which consider one aspect of the problem at a time. Initially, a single server (one delivery ship side) is considered. The order of the receivers alongside is arbitrarily fixed, and the optimal allocation of time alongside to each receiver is determined. The model is then extended to include the determination of the optimal sequence of receivers. That model is then extended to consider the optimal partitioning of receivers into sets assigned to each available delivery ship side.

1. Single Server, Fixed Receiver Sequence, Optimal Allocation

There are R receivers. Let the index j identify each receiver; $j = 1, \ldots, R$. Receiver j requests n_j lifts of ammunition.

Assumption 4.1: Each receiver gets at most one opportunity alongside the delivery ship.

Backward induction will be used to determine the allocation of time alongside the delivery ship to each receiver which yields the optimal expected combat value. Define the stages of the induction as the number of receivers remaining to be served, indexed by r; $r = 1, \ldots, R$. For notational convenience, let the index j which identifies each receiver correspond to the stage in which each is served (i.e., $j \equiv r$). For example, the first receiver to be served is identified by the index $j = R$ since there are R receivers remaining to be served including itself; and the last receiver to be served is identified by the index $j = 1$.

Assumption 4.2: Once alongside, the lifts are delivered to each receiver in a fixed sequence.

Let the index l define the fixed sequence in which lifts of possibly differing nature are transferred to each receiver; $l = 1, \ldots, n_j$.

Let $x_j(l)$ denote the elapsed time from the moment when receiver j commences replenishment until the delivery ship completes transferring the l^{th} lift.

Let $c_j(l)$ denote the elapsed time from the moment when receiver j commences replenishment until completion of strikedown of the l^{th} lift. Strikedown completion time,

$c_j(l)$, equals transfer completion time, $x_j(l)$, plus handling, waiting, and strikedown time on the receiving ship. These variables are depicted in Figure 8.

Although no assumptions are made concerning the initial transfer completion times, $x_j(1)$, it will typically include *setup* time for the receiving ship to maneuver alongside the delivery ship and connect transfer equipment.

Assumption 4.3: All receivers can be ready to start replenishing at the time designated for the replenishment to commence.

In the terminology of stochastic scheduling, Assumption 4.3 says that all jobs are *released* at the outset.

Let $v_j(l)$ denote the marginal combat value of the l^{th} lift delivered to receiver j. This marginal combat value accrues when strikedown is completed, if the raid arrival is later than strikedown completion.

Assumption 4.4: The quantities $v_j(l)$ may depend on the weapons state of receiver j, but are independent of the weapons states of the other receivers.

Assumption 4.5: The $v_j(l)$ are non-negative.

Assumption 4.6: The air raid interarrival time has an exponential distribution with a known mean.

Let the random variable T denote the air raid interarrival time, with mean τ ; and $\overline{F}(t) = P[T \geq t] = e^{-t/\tau}$.

Let the decision variable k_j denote the number of lifts to deliver to receiver j ; $k_j \in \{0,1, \ldots, n_j\}$. Then the time alongside allotted to receiver j is the transfer completion time, $x_j(k_j)$; where $x_j(0) \equiv 0$.

Let f_r denote the total expected combat value with r receivers remaining to be served, if the number of lifts allotted are $k_r, \ldots, k_2, k_1$, respectively, to each remaining receiver. Let f_r^* denote the maximal f_r if the optimal number of lifts, k_j^*, are allotted to each remaining receiver; $j = 1, \ldots, r$.

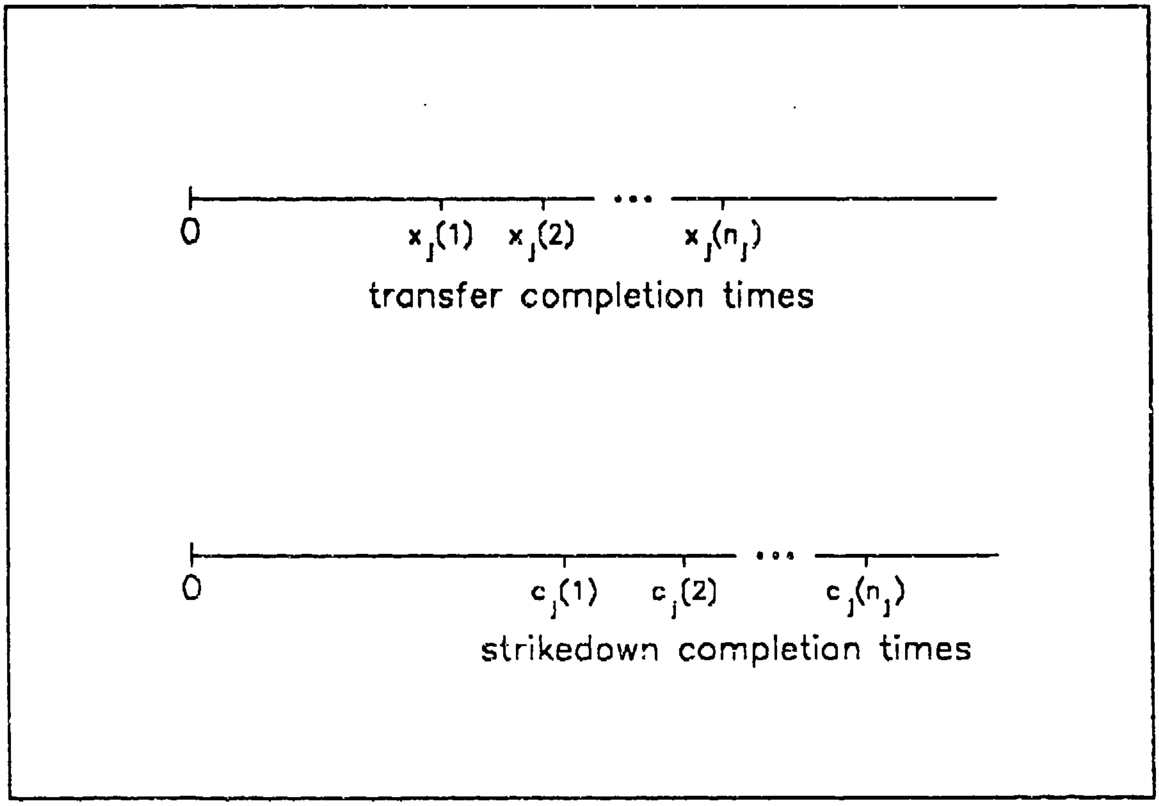


Figure 8. CONREP Process

Proposition 4.1: (a) Under Assumptions 4.1 through 4.6, for a single server, and a fixed sequence of receivers, the maximum expected combat value with r receivers remaining to be served, is

$$f_r^* = \max_{k=0,1,\ldots,n_r} \left[V_r(k) + \overline{F}(x_r(k))\, f_{r-1}^* \right] \quad ; \tag{4.1}$$

for $r = 2, \ldots, R$; and

$$f_1^* = V_1(n_1) \quad ; \tag{4.2}$$

where

$$V_r(k) \equiv \sum_{l=0}^{k} v_r(l)\, \overline{F}(c_r(l)) \quad ,$$

and where $v_r(0) \equiv 0$, and $c_r(0) \equiv 0$.

(b) The optimal decision at each stage (i.e., the optimal number of lifts to allot to receiver r), k_r^*, for $r = 2, \ldots, R$, is the argument which maximizes the functional equations (4.1); and $k_1^* = n_1$.

Proof 1 (Backward induction argument): With r receivers remaining to be served, the expected immediate return if the decision is to allot k lifts to the next receiver to be served, is the expected combat value contribution

$$V_r(k) = \sum_{l=0}^{k} v_r(l)\, \overline{F}(c_r(l)) \quad ,$$

where $v_r(0) \equiv 0$, and $c_r(0) \equiv 0$. Using the memoryless property of the exponential distribution, the expected future return is zero if the process is interrupted before transfers to the current receiver are completed, and is f_{r-1}^*, if the process is not interrupted and the remaining $r-1$ receivers are served optimally. If the decision is to allot k lifts to the next receiver to be served, then the probability that the process is not interrupted is the probability that the delivery of k lifts to receiver r is completed $\overline{F}(x_r(k))$. Hence Equation (4.1) follows from the classic backward induction argument. Equation (4.2), the initial condition, follows directly from Assumption 4.5. ∎

Proof 2 (A detailed derivation): The total expected combat value for the entire CONREP process can be expressed as

$$\begin{aligned} f_R &= v_R(1)\,\overline{F}(c_R(1)) + v_R(2)\,\overline{F}(c_R(2)) + \cdots + v_R(k_R)\,\overline{F}(c_R(k_R)) \\ &\quad + v_{R-1}(1)\,\overline{F}(c_{R-1}(1) + x_R(k_R)) + v_{R-1}(2)\,\overline{F}(c_{R-1}(2) + x_R(k_R)) \\ &\qquad + \cdots + v_{R-1}(k_{R-1})\;\overline{F}(c_{R-1}(k_{R-1}) + x_R(k_R)) \\ &\quad \vdots \\ &\quad + v_1(1)\,\overline{F}\left(c_1(1) + \sum_{j=2}^{R} x_j(k_j) \right) + v_1(2)\,\overline{F}\left(c_1(2) + \sum_{j=2}^{R} x_j(k_j) \right) \\ &\qquad + \cdots + v_{1k_1}\,\overline{F}\left(c_{1k_1} + \sum_{j=2}^{R} x_j(k_j) \right) \quad . \end{aligned} \tag{4.3}$$

Using the assumption of the exponential distribution, the $\overline{F}(\, x_\bullet(k_\bullet)\,)$ terms can be factored out, and Equation (4.3) can then be rewritten as

$$
\begin{aligned}
f_R &= [v_R(1)\,\overline{\mathrm{F}}(c_R(1)) + v_R(2)\,\overline{\mathrm{F}}(c_R(2)) + \cdots + v_R(k_R)\,\overline{\mathrm{F}}(c_R(k_R))] \\
&+ \overline{\mathrm{F}}(x_R(k_R))\,[v_{R-1}(1)\,\overline{\mathrm{F}}(c_{R-1}(1)) + v_{R-1}(2)\,\overline{\mathrm{F}}(c_{R-1}(2)) + \cdots + v_{R-1}(k_{R-1})\,\overline{\mathrm{F}}(c_{R-1}(k_{R-1}))] \\
&\vdots \\
&+ \prod_{j=2}^{R} \overline{\mathrm{F}}(x_j(k_j))\,[v_1(1)\,\overline{\mathrm{F}}(c_1(1)) + v_1(2)\,\overline{\mathrm{F}}(c_1(2)) + \cdots + v_1(k_1)\,\overline{\mathrm{F}}(c_1(k_1))] \ .
\end{aligned} \tag{4.4}
$$

Define the conditional expected combat value contribution of receiver j as

$$
V_j(k) = \sum_{l=0}^{k} v_j(l)\,\overline{\mathrm{F}}(c_j(l)) \quad ,
$$

where $v_j(0) \equiv 0$, and $c_j(0) \equiv 0$. Then Equation (4.4) can be rewritten as

$$
f_R = V_R(k_R) + \overline{\mathrm{F}}(x_R(k_R))\,V_{R-1}(k_{R-1}) + \cdots + \prod_{j=2}^{R} \overline{\mathrm{F}}(x_j(k_j))\,V_1(k_1) \quad . \tag{4.5}
$$

Or equivalently, as

$$
\begin{aligned}
f_R &= V_R(k_R) + \overline{\mathrm{F}}(x_R(k_R)) \\
&\times \Big[\, V_{R-1}(k_{R-1}) + \overline{\mathrm{F}}(x_{R-1}(k_{R-1}))\,\Big[\cdots \Big[\, V_2(k_2) + \overline{\mathrm{F}}(x_2(k_2))\,[\,V_1(k_1)\,]\,\Big] \cdots \Big]\,\Big] \ .
\end{aligned} \tag{4.6}
$$

Using the representation of Equation (4.6), it is seen that f_R can be maximized by successively maximizing terms in brackets, starting from the innermost pair. This is a backward induction on the number of receivers remaining to be served. Thus, the functional equation (4.1) has been obtained for $r = 2, \ldots, R$; and

$$
f_1^* = \max_{k=0,1,\ldots,n_1} [\,V_1(k)\,] \quad ; \tag{4.7}
$$

Since the $v_j(l)$ are assumed to be non-negative, then $V_1(k)$ is non-decreasing in the argument k, and thus f_1^* is obtained by setting k at its upper bound, n_1 (i.e., $k_1^* = n_1$); giving Equation (4.2). ■

2. Single Server, Optimal Receiver Sequence and Allocation

This model is now expanded to include the determination of the optimal sequence of receivers, as well as the optimal allocation of time alongside to each receiver.

Clearly, one approach to this optimization could be to use Equations (4.1) and (4.2) to evaluate f_R^* for the R factorial possible sequences in which the receivers can be scheduled alongside the delivery ship. Considering that a typical battle group has only six to eight receiving ships, total enumeration of all possible receiver sequences is computationally feasible. However, computational savings may be obtained by using a backward induction which *implicitly* enumerates all possible receiver sequences.

The stages of the induction are defined, as before, as the number of receivers remaining to be served, numbered with the index r.

Define *states* at each stage as the subsets of receivers remaining to be served (i.e., each state is a list of the identities of the receivers remaining to be served).[3] Let s denote such a state. For example, for $R = 3$, the possible states at stage 2 are $s = \{1,2\}$, $s = \{1,3\}$, and $s = \{2,3\}$. At stage r, there are $\binom{R}{r}$ such states. Let S_r denote the set of possible states at stage r. For example, for $R = 3$

$$
\begin{aligned}
S_1 &= \{\ \{1\},\{2\},\{3\}\ \} \\
S_2 &= \{\ \{1,2\},\{1,3\},\{2,3\}\ \} \\
S_3 &= \{\ \{1,2,3\}\ \}\ ;
\end{aligned}
$$

It is also convenient to use the set theory notation of *difference* or *relative complement*. Let $s\backslash\{j\}$ denote the set which contains the elements which belong to s but not including the element j. For example, if $s = \{1,2,3\}$, then

$$s\backslash\{2\} = \{1,3\}\ .$$

Functional equations can now be written to recursively solve for the sequence of receivers and allocations of time alongside which maximizes expected combat value. Let $f_r^*(s)$ denote the maximal expected combat value obtained by deciding on the optimal sequence of remaining receivers, and the optimal number of lifts to allot to each remaining receiver.

Proposition 4.2: (a) Under Assumptions 4.1 through 4.6, for a single server, the maximum expected combat value with r receivers remaining to be served, where the identities of the r receivers are the elements of the set s, is

---

[3] This use of the word *state* follows the classic terminology of dynamic programming as used, for example, by Bellman [Ref. 17]. This particular choice of state space used to formulate functional equations in this problem should not be confused with the *weapons state* used elsewhere in this work.

$$f_r^*(s) = \max_{j \in s} \left[\max_{k=0,1,\dots,n_j} \left[V_j(k) + \overline{F}(x_j(k))\, f_{r-1}^*(s \setminus \{j\}) \right] \right] \; ; \tag{4.8}$$

for all $s \in S_r$, and for $r = 2, \dots, R$; and

$$f_1^*(j) = V_j(n_j) \; ; \tag{4.9}$$

for $j = 1, 2, \dots, R$.

(b) The optimal decisions for each state at stages $r = 2, \dots, R$ are the arguments which maximize the functional equations (4.8). These decisions give the identity of the optimal receiver to schedule for service in that state at that stage, and the corresponding optimal number of lifts allotted to that receiver.

Proposition 4.2 conforms to the *principle of optimality* as given by Bellman [Ref. 17]:

> An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

Example 4.1. As an example of how the induction would proceed, for $R = 3$, Equations (4.8) and (4.9) would be expanded as follows:

$$\begin{aligned} f_1^*(1) &= V_1(n_1) \\ f_1^*(2) &= V_2(n_2) \\ f_1^*(3) &= V_3(n_3) \end{aligned}$$

$$f_2^*(1,2) = \max \begin{bmatrix} \max_{k=0,1,\dots,n_1} \left[V_1(k) + \overline{F}(x_1(k))\, f_1^*(2) \right] \\ \max_{k=0,1,\dots,n_2} \left[V_2(k) + \overline{F}(x_2(k))\, f_1^*(1) \right] \end{bmatrix}$$

$$f_2^*(1,3) = \max \begin{bmatrix} \max_{k=0,1,\dots,n_1} \left[V_1(k) + \overline{F}(x_1(k))\, f_1^*(3) \right] \\ \max_{k=0,1,\dots,n_3} \left[V_3(k) + \overline{F}(x_3(k))\, f_1^*(1) \right] \end{bmatrix}$$

$$f_2^*(2,3) = \max \begin{bmatrix} \max_{k=0,1,\dots,n_2} \left[V_2(k) + \overline{F}(x_2(k))\, f_1^*(3) \right] \\ \max_{k=0,1,\dots,n_3} \left[V_3(k) + \overline{F}(x_3(k))\, f_1^*(2) \right] \end{bmatrix}$$

$$f_3^*(1,2,3) = \max \begin{bmatrix} \max\limits_{k=0,1,\ldots,n_1} \left[V_1(k) + \overline{F}(x_1(k))\, f_2^*(2,3) \right] \\ \max\limits_{k=0,1,\ldots,n_2} \left[V_2(k) + \overline{F}(x_2(k))\, f_2^*(1,3) \right] \\ \max\limits_{k=0,1,\ldots,n_3} \left[V_3(k) + \overline{F}(x_3(k))\, f_2^*(1,2) \right] \end{bmatrix}$$

In this example, it is seen that, in the third stage, with receivers 1, 2, and 3 remaining to be served, the three factorial possible receiver sequences are *implicitly* enumerated by considering only the three cases of serving one receiver followed by serving the remaining two optimally.

The computational complexity of using the backward induction of Proposition 4.2 is of order $\bar{n}R^2 2^R$ versus total enumeration of all receiver sequences which is of order $\bar{n}R^2(R-1)!$; where $\bar{n}$ is the average number of lifts requested per receiver. Thus considerable computational savings are obtained for $R > 5$.

3. Two Servers, Optimal Partition, Sequence and Allocation

Besides computational improvement, another advantage of using Equations (4.8) and (4.9), is that the intermediate results, $f_r^*(s)$ can be directly applied to extend the model to two servers (i.e., parallel service at port and starboard delivery ship stations).

Let $\tilde{s}$ denote the *complement* of state s. For example, for $R = 3$, if $s = \{1,3\}$, then $\tilde{s} = \{2\}$; if $s = \{1,2,3\}$, then $\tilde{s} = \{0\}$; etc. Let P denote a partition of the receivers into the sets s and $\tilde{s}$.

Let $f(P^*)$ denote the maximum expected combat value which is attained with the optimal partition P^*.

Proposition 4.3: If $f_r^*(s)$ is obtained using Proposition 4.2 for a single server, then the optimization over all possible partitions of receivers between two servers can be written as

$$f(P^*) = \max_{\substack{r=0,\ldots,R \\ s\in S_r}} \left[f_r^*(s) + f_{R-r}^*(\tilde{s}) \right] \; ; \qquad (4.10)$$

where $f_0^*(0) \equiv 0$.

There are 2^R such partitions. However, since the partitions $\{s, \tilde{s}\}$ and $\{\tilde{s}, s\}$ are symmetric, only half that number, or 2^{R-1} partitions need be considered. Again, considering that a typical battle group has only six to eight receiving ships, 2^7 comparisons is computationally reasonable.

Example 4.1 (continued). Continuing the example for $R = 3$, Equation (4.10) expands as follows:

$$f(P^*) = \max \begin{bmatrix} f_3^*(1,2,3) \\ f_2^*(1,2) + f_1^*(3) \\ f_2^*(1,3) + f_1^*(2) \\ f_2^*(2,3) + f_1^*(1) \end{bmatrix} ;$$

where all $f_r^*(s)$, for $r < 3$ on the right hand side are intermediate results obtained in computing $f_3^*(1,2,3)$ using Proposition 4.2.

This procedure can be easily specialized to consider selected partitions if, for example, some receivers are restricted to a particular delivery ship side, such as in the case for an aircraft carrier which can only replenish from the port side of the delivery ship.

4. Computer Implementation

The dynamic programming recursions of Propositions 4.2 and 4.3 have been implemented in FORTRAN and run on an IBM 3033 computer at the Naval Postgraduate School. Sample problems with up to eight receiving ships in a Battle Group, requiring up to 50 lifts each, executed the recursion in less than two tenths of a second CPU time. Larger problems are computationally feasible, but not of practical interest in the context of a Battle Group.

A concise version of the program is listed in Appendix E. The following four ship example demonstrates the use of the program.

Example 4.2. This is an example of a CONREP scheduling problem for a small Battle Group with one delivery ship capable of providing connected replenishment on two sides, and four combatant ships. A summary of the ammunition requests is shown in Figure 9.

Receiver	Weapon	Capacity	Current State	Quantity Requested
Ship1	WepA	40	10	30
Ship1	WepD	4	1	3
Ship1	WepE	4	1	3
Ship1	WepF	20	15	5

		Ship1 total lifts req.		41
Ship2	WepB	20	5	15
Ship2	WepD	2	1	1
Ship2	WepE	2	1	1
Ship2	WepF	10	4	6

		Ship2 total lifts req.		23
Ship3	WepC	8	2	6
Ship3	WepD	2	0	2
Ship3	WepE	2	0	2
Ship3	WepF	20	10	10

		Ship3 total lifts req.		20
Ship4	WepC	4	0	4
Ship4	WepD	1	0	1
Ship4	WepE	2	0	2
Ship4	WepG	10	4	6

		Ship4 total lifts req.		13

Figure 9. Example 4.2 Summary of Ammunition Requests

The complete list of the individual ship requests by lift, giving the CONREP dynamic program inputs, including combat values, transfer completion times and strikedown completion times, are listed in Appendix E. The procedures described in Appendices B and C were used to generate that data.

The Resulting CONREP schedule for Example 4.2 is shown in Figure 10.

The Battle Group Commander is assured that this schedule maximizes the expected combat value of weapons strikedowns completed before the uncertain time at which the next raid arrives. However, summarizing the results another way provides a

Delivery side 1:

Receiver	Number of Lifts	Time Alongside
3	10	1.24
2	23	2.80

Delivery side 2:

Receiver	Number of Lifts	Time Alongside
4	7	0.88
1	41	4.96

Figure 10. Example 4.2 Resulting CONREP Schedule

better picture of what this schedule provides. Considering the times of strikedown completions, the projected weapons state of the Battle Group at the expected time of the next raid arrival is summarized in Figure 11. The numbers in parentheses show additional lifts for which the transfer is projected to be completed by E(T), but for which strikedown is not. This summary shows that although receivers 3 and 4 were cut off in the CONREP schedule, they were scheduled to receive a fair share. Also reflected in the summary, is a scheduling trade-off for receivers 1 and 2. Although each was scheduled last on their respective delivery side, so that at E(T) their weapons strikedown completion was less in percentage than the other receivers, there is good probability that they will be able to complete additional strikedowns

C. DYNAMIC SCHEDULE REVISION

In this section, the problem of dynamically revising the optimal receiver sequence, lift allocation, and partition between two servers is considered. Two distinct motivations for dynamic schedule revision arise in this problem -- new information and *release dates.*

The setting in which the first arises, is that after the process has been in progress for some time, new information may become available which suggests revising the schedule. One type of new information concerns the deterministic times to conduct transfers. These times are determined by parameters which may change (and can be observed). For example, a receiver in progress may have an equipment malfunction from which revised strikedown completion times may be obtained. Another type of new information concerns Assumption 4.6, that the air raid interarrival time has an exponential distrib-

Receiver	Weapon	Capacity	Projected State
Ship1	WepA	40	15 (+14)
Ship1	WepD	4	4
Ship1	WepE	4	4
Ship1	WepF	20	15
Ship2	WepB	20	15 (+5)
Ship2	WepD	2	2
Ship2	WepE	2	2
Ship2	WepF	10	7 (+2)
Ship3	WepC	8	8
Ship3	WepD	2	2
Ship3	WepE	2	2
Ship3	WepF	20	10
Ship4	WepC	4	4
Ship4	WepD	1	1
Ship4	WepE	2	2
Ship4	WepG	10	6

Figure 11. Example 4.2 Projected Weapons States at E(T).

ution with a known mean. Of course the mean will not truly be *known*, but rather estimated (perhaps by intelligence analysts). As time goes by, the Battle Group may get a revised estimate of the expected air raid arrival time, which should be used to revise the schedule.

The term *dynamic* is used to capture the idea that the revision takes place while the process is in progress, and takes into account new information as it becomes available. It should be noted, however, that the revised schedule will be a new *static list* which specifies the remaining process given the information available at the time of the revision.

The second motivation for dynamic schedule revision relates to the property that it provides a new static list, and concerns Assumption 4.3 -- that all receivers can be ready to start replenishing at the time designated for the replenishment to commence (i.e., that all jobs are released at time zero). In the combat CONREP setting, it is common that the receivers would arrive at staggered times (which would be known). In the terminology of scheduling theory, these would be deterministic *release dates* of jobs. An intuitive argument for developing an approach to the release date problem using dynamic sched-

ule revision, is that up until the moment when a new receiver arrives, the stochastic time available should be used optimally for the receivers who are present (i.e., an optimal initial schedule), and then, if a raid has not terminated the process before the arrival of a new receiver, use the remaining stochastic time optimally for the receivers present including the new arrival (i.e., dynamic revision).

The dynamic schedule revision approach is developed in steps starting with some special cases.

1. Simple Revision

Propositions 4.2 and 4.3 will be modified to provide for revising the optimal lift allocation for the receiver(s) in progress, and for the receivers who have not yet started service, revising the optimal sequence, the optimal lift allocation, and the optimal partition between two servers.

This revision is called *simple* because none of the assumptions previously made will change. In particular, Assumption 4.1, that each receiver gets at most one opportunity alongside the delivery ship, will continue to be a condition of *simple* revision. Also, any continuing service to a receiver in progress, is constrained to follow immediately before any service to the remaining receivers. The model following this one will consider interrupting and rescheduling a receiver in progress.

Additional notation is now introduced to describe the process from the moment when a simple schedule revision is made. Let ja denote the identity of the receiver in progress at server a. Double character variable names are used to avoid an additional level of subscripting. The second character, a , represents a letter designation for the server.

Example 4.3. In the context of this problem, where the servers are the sides of the delivery ship, the receivers in progress are denoted jp and js to represent the *port* and *starboard* sides, respectively.

The identification of a receiver in progress, ja, can be null if the server is idle.

For receiver ja, let l_a denote the index number of the lift in the process of being transferred. If a revision is to be made at the moment when a transfer to receiver ja is completed, then l_a is defined to be the index number of the transfer just completed to receiver ja . Define *remaining* transfer completion and strikedown completion times for receiver ja as $\hat{x}_a(l) = x_{ja}(l) - t_a$, and $\hat{c}_a(l) = c_{ja}(l) - t_a$, for $l = l_a, \ldots , n_{ja}$; where t_a is the elapsed

time since receiver ja started service. Also, define the *remaining* expected combat value contribution of receiver ja as

$$\hat{V}_a(k) = \sum_{l=l_a}^{k} v_{ja}(l)\,\bar{F}(c_a(l)) \quad .$$

Let S' denote the set of receivers who have not yet started service, and let R' denote the total number of receivers in S'. Also let S'' denote the set of receivers in progress, and let R'' denote the number of receivers in S''. In the context of this problem, $R'' \in \{0, 1, 2\}$.

The stages of the induction are defined, as before, as the number of receivers remaining to be served, numbered with the index r; and states are defined, as before, as the subsets of receivers remaining to be served. Here, however, it is convenient to distinguish the intermediate states in stages 1 through R' which include only those receivers who have not yet started service. Let $s' \subset S'$ denote an intermediate state. An additional *final* state can be defined for each server in stages 2 through $R'+1$ by adding the receiver in progress to an intermediate state in the previous stage. Using the set theory notation for *union*, let $s' \bigcup \{ja\}$ denote such a state. In stage 1, the final state for server a is $\{ja\}$. As before, let S_r denote the set of possible states at stage r.

Example 4.3 (continued). If there were initially $R = 5$ receivers, and under the initial schedule, receiver 3 has completed service and the receivers in progress are $jp = 2$ and $js = 5$, then $S'' = \{2,5\}$, $R'' = 2$, and the set of receivers who have not yet started service is $S' = \{1,4\}$, and $R' = 2$. There are two possible intermediate states at stage 1 (one receiver to be served) which are $s' = \{1\}$ and $s' = \{4\}$. The possible final states at stage 2 for the port side of the delivery ship are $s' \bigcup \{jp\} = \{1,2\}$, and $s' \bigcup \{jp\} = \{4,2\}$, since the final state for that side includes the receiver in progress (and any continuing service to that receiver follows immediately before any service to the remaining receivers). Also, the possible final states at stage 2 for the starboard side of the delivery ship are $s' \bigcup \{js\} = \{1,5\}$, and $s' \bigcup \{js\} = \{4,5\}$. All of the possible states are shown in

Table 5.

Table 5. EXAMPLE 4.3 POSSIBLE STATES

Stage (r)	**Possible States (S_r)**		
	Intermediate states (s')	**Final states**	
		Port	**Starboard**
1	{1} , {4}	{2}	{5}
2	{1,4}	{1,2} , {4,2}	{1,5} , {4,5}
3	{0}	{1,4,2}	{1,4,5}

Functional equations can now be written to find the optimal schedule revision when there is at least one service in progress.

Proposition 4.4: Under Assumptions 4.1 through 4.6,

(a) The maximum expected combat value with 1 receiver remaining to be served, is

$$f_1^*(j) = V_j(n_j) \quad ; \tag{4.11}$$

for $j \in \{S' \cup S''\}$;

(b) For intermediate states s', the maximum expected combat value with r receivers, who have not yet started service, remaining to be served, where the identities of the r receivers are the elements of the set s', is

$$f_r^*(s') = \max_{j \in s'} \left[\max_{k=0,1,\dots,n_j} \left[V_j(k) + \overline{F}(x_j(k))\, f_{r-1}^*(s' \backslash \{j\}) \right] \right] \quad ; \tag{4.12}$$

for all $s' \in S_r$, and for $r = 2, \dots, R'$;

(c) The optimal decisions for each intermediate state s' at each stage are the arguments which maximize the functional equations (4.12). These decisions give the identity of the optimal receiver to schedule for service in that state at that stage, and the corresponding optimal number of lifts allotted to that receiver;

(d) For final states which include each receiver in progress, ja , the maximum expected combat value with $r + 1$ receivers remaining to be served, where the identities of the $r + 1$ receivers are the elements of the set $s' \cup \{ja\}$, is

$$f_{r+1}{}^*(s' \cup \{ja\}) = \max_{k=l_a, \ldots, n_{ja}} \left[\hat{V}_a(k) + \overline{F}(\hat{x}_a(k)) \, f_r{}^*(s') \right] \; ; \tag{4.13}$$

for all $s' \in S_r$, and for $r = 1, \ldots, R'$;

(e) The optimal decision for each final state $s' \cup \{ja\}$ at each stage is the argument which maximizes the functional equations (4.13). This decision gives the lift number after which service should be terminated for the receiver in progress, ja;

(f) The optimization over all possible partitions of receivers between two servers, designated a and b, can be written as

$$f(P^*) = \max_{\substack{r \\ s' \in S_r}} \left[f_r{}^*(s' \cup \{ja\}) + f_{R'+R''-r}{}^*(\tilde{s}' \cup \{jb\}) \right] \; . \tag{4.14}$$

If ja is not null, then $r \in \{1, \ldots, R' + 1\}$. If ja is null, then $r \in \{0, \ldots, R'\}$. When either ja or jb is null, define $f_0{}^*(0) \equiv 0$.

Discussion. Equation (4.11) follows directly from Proposition 4.2. Equations (4.12) and (4.13) are specializations of Equation (4.8) in Proposition 4.2 which consider, respectively, states which exclude or include a receiver in progress. Equation (4.12), the case where receivers in progress are excluded, is a direct application of Equation (4.8), where s' replaces s. Equation (4.13), the case where receivers in progress are included, is an adaptation of Proposition 4.1 in which the order of receivers was specified. This follows because when considering a state which includes the receiver in progress, any continuing service to that receiver follows immediately before any service to the remaining receivers. In Equation (4.13) the decision variable k, which gives the total number of lifts allocated to that receiver, is limited to take on values from l_a to n_{ja}, since the receiver in progress, ja, has lift number l_a in progress. Equation (4.14) is a specialization of Proposition 4.3 where the partition of receivers between the two servers is

limited due to the receiver(s) in progress being constrained to continue service from the corresponding server.

Example 4.3 (continued). To show how the induction would proceed, Equations (4.11) through (4.14) would be expanded as follows:

$$\begin{aligned} f_1^*(1) &= V_1(n_1) \\ f_1^*(2) &= V_2(n_2) \\ f_1^*(4) &= V_4(n_4) \\ f_1^*(5) &= V_5(n_5) \end{aligned}$$

$$f_2^*(1,4) = \max \begin{bmatrix} \max\limits_{k=0,1,\ldots,n_1} \left[V_1(k) + \overline{\mathrm{F}}(x_1(k))\, f_1^*(4) \right] \\ \max\limits_{k=0,1,\ldots,n_4} \left[V_4(k) + \overline{\mathrm{F}}(x_4(k))\, f_1^*(1) \right] \end{bmatrix}$$

$$f_2^*(1,2) = \max_{k=l_p,\ldots,n_2} \left[\hat{V}_p(k) + \overline{\mathrm{F}}(\hat{x}_p(k))\, f_1^*(1) \right]$$

$$f_2^*(4,2) = \max_{k=l_p,\ldots,n_2} \left[\hat{V}_p(k) + \overline{\mathrm{F}}(\hat{x}_p(k))\, f_1^*(4) \right]$$

$$f_2^*(1,5) = \max_{k=l_s,\ldots,n_5} \left[\hat{V}_s(k) + \overline{\mathrm{F}}(\hat{x}_s(k))\, f_1^*(1) \right]$$

$$f_2^*(4,5) = \max_{k=l_s,\ldots,n_5} \left[\hat{V}_s(k) + \overline{\mathrm{F}}(\hat{x}_s(k))\, f_1^*(4) \right]$$

$$f_3^*(1,4,2) = \max_{k=l_p,\ldots,n_2} \left[\hat{V}_p(k) + \overline{\mathrm{F}}(\hat{x}_p(k))\, f_2^*(1,4) \right]$$

$$f_3^*(1,4,5) = \max_{k=l_s,\ldots,n_5} \left[\hat{V}_s(k) + \overline{\mathrm{F}}(\hat{x}_s(k))\, f_2^*(1,4) \right]$$

$$f(P^*) = \max \begin{bmatrix} f_3^*(1,4,2) + f_1^*(5) \\ f_2^*(1,2) + f_2^*(4,5) \\ f_2^*(4,2) + f_2^*(1,5) \\ f_1^*(2) + f_3^*(1,4,5) \end{bmatrix} .$$

When there are no receivers in progress (i.e., jp and js both null), then $S'' = \{0\}$, $R'' = 0$, and Proposition 4.4 reduces to Propositions 4.2 and 4.3 for the receivers who have not yet started service.

2. Interrupting a Receiver in Progress and Rescheduling

In this section, the assumption that each receiver gets at most one opportunity alongside the delivery ship (Assumption 4.1) is relaxed to permit the service to a receiver in progress to be interrupted, and to allow all receivers' unfilled requests to be considered for additional service following the interruption.

Assumption 4.7: When a schedule is revised, independent of past service, and service in progress, each receiver gets at most one future opportunity alongside the delivery ship.

The *simple revision* considered in the previous section is a special case of revision with interruption and rescheduling in two respects. Firstly, in simple revision, receivers who previously completed service were not considered eligible for additional service. And secondly, because simple revision can be thought of as interrupting a receiver in progress, and rescheduling it for additional service constrained to commence immediately. In order to generalize the second idea and allow the additional service to commence after some intervening service to other receivers, any effect that the interruption delay has on transfer and strikedown completion times must be considered. For an initial approach to this problem, a simplifying assumption will be made.

Assumption 4.8: The receiver processing times $c_j(l)$ -- $x_j(l)$ do not depend on the sequence in which lifts are transferred or on the time of transfer.

In the context of this problem, for Assumption 4.8 to be valid, strikedown queues can not develop, and, consequently, any interruption delay will not affect strikedown completion times. In contrast to this, if strikedown queues did develop, the receiver processing times would include waiting time in the strikedown queue. Then an interruption delay would allow the strikedown queue to shorten (or empty), and hence reduce the receiver processing times following a delay. Furthermore, the amount by which the strikedown times are shortened would depend on the length of the delay, which would

not be not known until after rescheduling is finished. The problem with strikedown queues is left for future work.

Returning to the idea that simple revision can be thought of as interrupting the receiver in progress, and rescheduling it for additional service constrained to commence immediately, the formulation of the current problem of revision with interruption and rescheduling will use the following conceptualization:

- Any receivers in progress are interrupted as soon as the lift in the process of being transferred is completed;
- The remaining lifts requested are considered for rescheduling ;
- If after rescheduling, the additional service follows intervening receivers, or is shifted to another server, then the transfer time for the first additional job will typically include some additional setup time.
- If, however, the additional service commences immediately with the same server (as in the case of simple revision), then no additional setup time will be incurred.

Example 4.4. Suppose there are a total of $R = 3$ receivers, and receivers 2 and 3 are in progress at the delivery ship's port and starboard sides, respectively, and receiver 1 is ready to start service. Since the schedule is now to be revised, it is immaterial for which side receiver 1 had been previously scheduled, and immaterial whether or not receiver 1 had received previous service. A possible revised receiver sequence for both sides is:

Port side:	Receiver 2 (*without* additional setup time), followed by
	Receiver 3 (*with* additional setup time).
Starboard side:	Receiver 1.

Another possible revised receiver sequence for both sides is:

Port side:	Receiver 1, followed by
	Receiver 2 (*with* additional setup time).
Starboard side:	Receiver 3 (*without* additional setup time).

The inclusion or exclusion of additional setup time can be thought of as distinct initial conditions depending on when additional service for a receiver in progress commences. To accommodate this, each receiver in progress will be represented by a pair of *alternative pseudo-receivers*, each with a unique sequence of transfer completion and strikedown times that reflect the appropriate initial condition. As before, let ja denote the identity of the receiver in progress at server a. If the additional service for this receiver in progress is allowed to commence immediately with the same server, then ja will be represented as the pseudo-receiver ja_0. Otherwise, if the additional service for this receiver in progress is required to follow intervening receivers, or shifted to another server, then ja will be represented as the pseudo-receiver ja_1. The pair ja_0 and ja_1 are called *alternative* pseudo-receivers since any revised schedule will include one or the other. If referring to additional service regardless of when it commences, the alternative pseudo-receivers will be collectively denoted

$$ja_\bullet = (ja_0 \oplus ja_1) \ ;$$

where the $\oplus$ symbol is the logical *exclusive or* operator, which in this context may be read as *either* ja_0 *or* ja_1 *but not both.*

Example 4.4 (continued). In a manner similar to Example 4.3, receivers in progress at the port and starboard sides of the delivery ship are denoted jp and js, respectively. Considering alternative initial conditions for remaining service, the receiver in progress on the port side, jp , will be represented as either the pseudo-receiver jp_0, the pseudo-receiver jp_1, or collectively as $jp_\bullet$. Similarly, js will be represented as either the pseudo-receiver js_0, the pseudo-receiver js_1, or collectively as $js_\bullet$. In this example, $jp = 2$ and $js = 3$. Numerical examples for jp_0 , jp_1, js_0, js_1, $jp_\bullet$, and $js_\bullet$ will be given following some additional discussion.

For all receivers, let $\hat{n}_j$ denote the number of remaining lifts requested by receiver j. In terms of given data and the observed process up to the revision time, $\hat{n}_j$ is equal to the original number of lifts requested, n_j, minus lifts in progress and lifts previously delivered. In particular, the number of remaining lifts requested by receiver j are as follows:

- If receiver j has not yet started any service, then $\hat{n}_j = n_j$, the original number of lifts requested;

- If the receiver in progress is denoted ja, and l_a denotes the index number of the lift in the process of being transferred, then $\hat{n}_{ja} = n_{ja} - l_a$;
- And if receiver j has already completed previously scheduled service, and the decision variable k_j denotes the number of lifts previously delivered to receiver j, then $\hat{n}_j = n_j - k_j$.

Let the index l and the decision variable k now denote lifts *in addition to* lifts in progress and lifts previously delivered. In particular, let the index l define the fixed sequence in which *remaining* lifts are transferred to each receiver; $l = 1, \dots, \hat{n}_j$; and let the decision variable k_j denote the number of *additional* lifts to deliver to receiver j; $k_j \in \{0, 1, \dots, \hat{n}_j\}$. It should be noted that this use of indices for receivers in progress is a departure from the use in the previous section describing simple revision. In that special case, the original number of requests, n_j, and original indexing of lifts, $l = 1, \dots, n_j$ were used; and the number of lifts to deliver to the receiver in progress was simply revised, where the possible revised values were $k_{ja} \in \{l_a, \dots, n_{ja}\}$. In the more general case of revision with rescheduling, where for a receiver in progress $\hat{n}_{ja} = n_{ja} - l_a$, then the indexing of lifts and possible values of the decision variable are shifted back to their respective origins. It should be further noted that under this revised indexing of lifts, the nature (type of ammunition) of the l^{th} lift will, in general, be different than under the original indexing. Consequently, the transfer completion times, $x_j(l)$, strikedown completion times, $c_j(l)$, and the marginal combat values, $v_j(l)$ will have revised given values based on the current identity of the l^{th} lift.

For all receivers not in progress (i.e., $j \neq ja$), and for receivers in progress who may get rescheduled following some intervening service (i.e., $j = ja_1$), the original definitions of transfer and strikedown completion times apply to remaining service (even if the values are revised). This includes the provision that $x_j(1)$, the time it takes to transfer the first *remaining* lift, typically includes setup time. In contrast to this, for a receiver in progress who may get rescheduled to commence additional service immediately (i.e., $j = ja_0$), define $x_{ja_0}(l)$ and $c_{ja_0}(l)$, respectively, as transfer completion and strikedown completion times that exclude setup time.

Now, considering the nature of transfer and strikedown completion times, a convention may be adopted to assign numerical value for jp_0, jp_1, js_0, and js_1. Since the nature of transfer and strikedown completion times for ja_1 are the same as the original transfer and strikedown times for ja, it is convenient to re-use the receiver index by setting $ja_1 = ja$. However, since the nature of transfer and strikedown completion times for ja_0 exclude setup time, distinguishing indices should be used for these special pseudo-

receivers. Since there are R receivers, it is convenient to set $js_0 = R + 1$, and $jp_0 = R + 2$.

Example 4.4 (continued). In this example, set

$$\begin{aligned} jp_1 &= jp = 2 \\ js_1 &= js = 3 \end{aligned} .$$

And since $R = 3$, set

$$\begin{aligned} js_0 &= 4 \\ jp_0 &= 5 \end{aligned} .$$

Using the collective notation,

$$\begin{aligned} jp_\bullet &= (jp_0 \oplus jp_1) = (2 \oplus 5) \\ js_\bullet &= (js_0 \oplus js_1) = (3 \oplus 4) \end{aligned} .$$

It will be convenient to use additional notation to represent some special states. With the convention of using original index numbers for all ja_1, let S° denote the set of receivers and or pseudo-receivers identified by their original index $j \in \{1, \ldots, R\}$; and let s° denote any subset of S°. Thus s° is a state that excludes all ja_0, but may include any ja_1. Also let S' denote the set of receivers who are not currently being served; and let s' denote any subset of S'. Thus s' is a state that excludes all ja_0 and ja_1. If there are not receivers other than those in progress, then S' is the null set. These special states are adaptations of what was called an intermediate state in the previous section describing simple revision. In all cases, these special states consist of receivers who are to be scheduled for service which is not constrained to commence immediately. In the previous case of simple revision, s' could contain only receivers who had not yet started service, which were all *remaining* receivers other than those in progress. In the current case of interruption and rescheduling, s' is redefined, slightly, as a state which could contain *all* receivers other than those in progress, regardless of whether or not they received prior service; and s° is defined as a state which could contain all receivers other than those in progress, regardless of whether or not they received prior service, *as well as* the pseudo-receivers ja_1 who are not constrained to commence service immediately.

Example 4.4 (continued). In this example, the set of receivers and/or pseudo-receivers identified by their original index are

$$S^\circ = \{1, 2, 3\} \;\; ;$$

and the possible *original-index* states, including a null state are

$$s^\circ \in \Big\{ \{0\}\,,\ \{1\}\,,\ \{2\}\,,\ \{3\}\,,\{1{,}2\}\,,\ \{1{,}3\}\,,\ \{2{,}3\}\,,\ \{1{,}2{,}3\} \Big\} \;\; .$$

Also, the set of receivers who are not currently being served is

$$S' = \{1\} \;\; ;$$

and the possible *not-in-progress* states, including a null state are

$$s' \in \{\ \{0\}\,,\ \{1\}\ \} \;\; .$$

Functional equations can now be written to find the optimal schedule revision when service in progress is interrupted and all receivers' unfilled requests are considered for additional service following the interruption.

Proposition 4.5: Under Assumptions 4.2 through 4.8,

(a) The maximum expected combat value with 1 receiver remaining to be served, is

$$f_1{}^*(j) = V_j(\hat{n}_j) \;\; ; \tag{4.15}$$

for all j;

(b) For all original-index states s°, the maximum expected combat value with $r + 1$ receivers remaining to be served, where the identities of the $r + 1$ receivers are the elements of the set s°, is

$$f_{r+1}{}^*(s^\circ) = \max_{j \in s^\circ} \left[\max_{k=0,1,\dots,\hat{n}_j} \Big[V_j(k) + \overline{F}(x_j(k))\, f_r{}^*(s^\circ \backslash \{j\}) \Big] \right] \tag{4.16}$$

for all $s^\circ \in S_{r+1}$; and for $r = 1, \dots, R-2$;

(c) For server a, for particular original-index states s° such that $ja_1 \notin s^\circ$, the maximum expected combat value with $r + 1$ receivers remaining to be served, where the identities of the r receivers other than ja_1 are the elements of the set s°, is

$$f_{r+1}^*(s^\circ \cup \{ja_\bullet\}) = \max \left[f_{r+1}^*(s^\circ \cup \{ja_0\}),\, f_{r+1}^a(s^\circ \cup \{ja_1\}) \right] ; \tag{4.17}$$

where

$$f_{r+1}^*(s^\circ \cup \{ja_0\}) = \max_{k=0,\dots,\hat{n}_{ja0}} \left[V_{ja0}(k) + \overline{F}(x_{ja0}(k))\, f_r^*(s^\circ) \right] , \tag{4.18}$$

and

$$f_{r+1}^a(s^\circ \cup \{ja_1\}) = \max_{j \neq ja_1} \left[\max_{k=0,1,\dots,\hat{n}_j} \left[V_j(k) + \overline{F}(x_j(k))\, f_r^*(\{ja_1\} \cup s^\circ \setminus \{j\}) \right] \right] \tag{4.19}$$

for all s° such that $s^\circ \in S_r$ and $ja_1 \notin s^\circ$; and for $r = 1, \dots, R-1$;

(d) The optimal decisions for each state at each stage are the arguments which maximize the functional equations (4.16) through (4.19). These decisions give the identity of the optimal receiver to schedule for service in that state at that stage, and the corresponding optimal number of lifts allotted to that receiver;

(e) For all not-in-progress states s', the optimization over all possible partitions of receivers between two servers, designated a and b, can be written as

$$f(P^*) = \max_{\substack{r=0,\dots,R-2 \\ s' \in S_r}} \begin{bmatrix} \left[f_{r+2}^*(s' \cup \{ja_\bullet\} \cup \{jb_1\}) + f_{R-r-2}^*(\tilde{s}') \right] \\ \left[f_{r+1}^*(s' \cup \{ja_\bullet\}) + f_{R-r-1}^*(\tilde{s}' \cup \{jb_\bullet\}) \right] \\ \left[f_{r+1}^*(s' \cup \{jb_1\}) + f_{R-r-1}^*(\tilde{s}' \cup \{ja_1\}) \right] \\ \left[f_r^*(s') + f_{R-r}^*(\tilde{s}' \cup \{jb_\bullet\} \cup \{ja_1\}) \right] \end{bmatrix} ; \tag{4.20}$$

where $f_0^*(0) \equiv 0$.

Discussion. Equations (4.15) and (4.16) follow directly from Equations (4.11) and (4.12) in Proposition 4.4, with $\hat{n}_j$ replacing n_j, and s° replacing s'. The original-index states in Equation (4.16) may be *intermediate states* for any server. However, since pseudo-receiver ja_1 may be served first in any of these stages, these original-index states may be *final states* only for a server other than server a. Consequently, Equation (4.16) enters the recursion through the next to the last stage (i.e., to obtain $f_{R-1}^*(s^\circ)$). Equations (4.17), (4.18), and (4.19) collectively give the maximum expected value for *final states* for each server in which that server *either* serves the pseudo-receiver ja_0 who is constrained to commence service immediately, *or* serves the pseudo-receiver ja_1 who is constrained to commence service following some intervening receiver. Equation

(4.18), which follows directly from Equation (4.13) in Proposition 4.4, gives the maximum expected value for the *final state* which includes the pseudo-receiver ja_0 who is constrained to commence service immediately. Equation (4.19) is a specialization of Equation (4.16) in which the possibility of the pseudo-receiver ja_1 getting service in that stage is precluded (i.e., the outer maximization in Equation (4.19) is over $j \neq ja_1$). The superscript a on $f^a_{r-1}(s^\circ \cup \{ja_1\})$ is used to distinguish the server and the corresponding pseudo-receiver ja_1 who is precluded from immediate service. It may be noted (and exploited in implementation) that except for the final stage, and except for the precluded ordering, the computation on the right hand side of Equation (4.19) is obtainable from Equation (4.16). Equation (4.20) is a specialization of Proposition 4.3 where the partition of receivers and pseudo-receivers between the two servers is characterized by three special cases. (Four cases are used in Equation (4.20), but the fourth is simply a symmetric version of the first.) The first (or fourth) special case is where one server provides subsequent service to both the receiver who was in progress with this server and the receiver who was in progress with the other server; and the other server provides subsequent service to only receivers who were not in progress. The second special case is where both servers provide subsequent service to the receiver who was in progress with that server. And the third special case is where both servers provide subsequent service to the receiver who was in progress with the opposite server.

Example 4.4 (continued). To show how the induction would proceed, Equations (4.15) through (4.20) would be expanded as follows:

First Stage (Eq. 4.15):

$$
\begin{aligned}
f_1^*(1) &= V_1(\hat{n}_1)\\
f_1^*(2) &= V_2(\hat{n}_2)\\
f_1^*(3) &= V_3(\hat{n}_3)\\
f_1^*(4) &= V_4(\hat{n}_4)\\
f_1^*(5) &= V_5(\hat{n}_5)
\end{aligned}
$$

Second Stage (Eq. (4.16)):

$$f_2^*(1,2) = \max \begin{bmatrix} \max\limits_{k=0,1,\dots,\hat{n}_1} \left[V_1(k) + \overline{F}(x_1(k))\, f_1^*(2) \right] \\ \max\limits_{k=0,1,\dots,\hat{n}_2} \left[V_2(k) + \overline{F}(x_2(k))\, f_1^*(1) \right] \end{bmatrix}$$

$$f_2^*(1,3) = \max \begin{bmatrix} \max\limits_{k=0,1,\dots,\hat{n}_1} \left[V_1(k) + \overline{F}(x_1(k))\, f_1^*(3) \right] \\ \max\limits_{k=0,1,\dots,\hat{n}_3} \left[V_3(k) + \overline{F}(x_3(k))\, f_1^*(1) \right] \end{bmatrix}$$

$$f_2^*(2,3) = \max \begin{bmatrix} \max\limits_{k=0,1,\dots,\hat{n}_2} \left[V_2(k) + \overline{F}(x_2(k))\, f_1^*(3) \right] \\ \max\limits_{k=0,1,\dots,\hat{n}_3} \left[V_3(k) + \overline{F}(x_3(k))\, f_1^*(2) \right] \end{bmatrix}$$

Second Stage (Eq. (4.17) - (4.19)):

Starboard Server:

(Eq. (4.18)):

$$f_2^*(1,4) = \max_{k=0,1,\dots,\hat{n}_4} \left[V_4(k) + \overline{F}(x_4(k))\, f_1^*(1) \right]$$

$$f_2^*(2,4) = \max_{k=0,1,\dots,\hat{n}_4} \left[V_4(k) + \overline{F}(x_4(k))\, f_1^*(2) \right]$$

(Eq. (4.19)):

$$f_2^s(1,3) = \max_{k=0,1,\dots,\hat{n}_1} \left[V_1(k) + \overline{F}(x_1(k))\, f_1^*(3) \right]$$

$$f_2^s(2,3) = \max_{k=0,1,\dots,\hat{n}_2} \left[V_2(k) + \overline{F}(x_2(k))\, f_1^*(3) \right]$$

(Eq. (4.17)):

$$f_2^*(1,(3\oplus 4)) = \max\left[f_2^*(1,4)\,,\, f_2^s(1,3) \right]$$

$$f_2^*(2,(3\oplus 4)) = \max\left[f_2^*(2,4)\,,\, f_2^s(2,3) \right]$$

Port Server:

(Eq. (4.18)):

$$f_2^*(1,5) = \max_{k=0,1,\ldots,\hat{n}_5} \left[V_5(k) + \overline{F}(x_5(k))\, f_1^*(1) \right]$$

$$f_2^*(3,5) = \max_{k=0,1,\ldots,\hat{n}_5} \left[V_5(k) + \overline{F}(x_5(k))\, f_1^*(3) \right]$$

(Eq. (4.19)):

$$f_2^P(1,2) = \max_{k=0,1,\ldots,\hat{n}_1} \left[V_1(k) + \overline{F}(x_1(k))\, f_1^*(2) \right]$$

$$f_2^P(3,2) = \max_{k=0,1,\ldots,\hat{n}_3} \left[V_3(k) + \overline{F}(x_3(k))\, f_1^*(2) \right]$$

(Eq. (4.17)):

$$f_2^*(1,(2\oplus 5)) = \max\left[f_2^*(1,5)\,,\, f_2^P(1,2) \right]$$

$$f_2^*(3,(2\oplus 5)) = \max\left[f_2^*(3,5)\,,\, f_2^P(2,3) \right]$$

Third Stage (Eq. (4.17) - (4.19)):

Starboard Server:

(Eq. (4.18)):

$$f_3^*(1,2,4) = \max_{k=0,1,\ldots,\hat{n}_4} \left[V_4(k) + \overline{F}(x_4(k))\, f_2^*(1,2) \right]$$

(Eq. (4.19)):

$$f_3^s(1,2,3) = \max \begin{bmatrix} \displaystyle\max_{k=0,1,\ldots,\hat{n}_1} \left[V_1(k) + \overline{F}(x_1(k))\, f_2^*(2,3) \right] \\ \displaystyle\max_{k=0,1,\ldots,\hat{n}_2} \left[V_2(k) + \overline{F}(x_2(k))\, f_2^*(1,3) \right] \end{bmatrix}$$

(Eq. (4.17)):

$$f_3^*(1,2,(3\oplus 4)) = \max\left[f_3^*(1,2,4)\,,\, f_3^s(1,2,3) \right]$$

Port Server:

(Eq. (4.18)):

$$f_3^*(1,3,5) = \max_{k=0,1,\ldots,\hat{n}_5} \left[V_5(k) + \overline{F}(x_5(k))\, f_2^*(1,3) \right]$$

(Eq. (4.19)):

$$f_3^P(1,3,2) = \max \begin{bmatrix} \max\limits_{k=0,1,\ldots,\hat{n}_1} \left[V_1(k) + \overline{F}(x_1(k))\, f_2^*(2,3) \right] \\ \max\limits_{k=0,1,\ldots,\hat{n}_3} \left[V_3(k) + \overline{F}(x_3(k))\, f_2^*(1,2) \right] \end{bmatrix}$$

(Eq. (4.17)):

$$f_3^*(1,3,(2\oplus5)) = \max\left[f_3^*(1,3,5)\,,\, f_3^P(1,3,2) \right]$$

Partition (Eq. (4.20)):

$$f(P^*) = \max \begin{bmatrix} f_3^*(1,3,(2\oplus5)) \\ f_2^*(1,(2\oplus5)) + f_1^*(4) \\ f_2^*(3,(2\oplus5)) + f_1^*(1) \\ f_2^*(1,3) + f_1^*(2) \\ f_1^*(5) + f_2^*(1,(3\oplus4)) \\ f_1^*(1) + f_2^*(2,(3\oplus4)) \\ f_1^*(3) + f_2^*(1,2) \\ f_3^*(1,2,(3\oplus4)) \end{bmatrix} .$$

Further work with the combat CONREP problem, and its interaction with the combat VERTREP problem is discussed in the conclusions.

V. COMBAT SUPPORT LOGISTICS: SERVICE POLICIES WITH QUEUE LENGTH INFLUENCE

A. INTRODUCTION

A logistics problem faced by an operational unit, such as a deployed detachment of aircraft, involves setting a maintenance policy for organizational level repair of mission-essential components. For full combat mission capability, an aircraft must have several different major avionics components available. Let the index i identify each of the different types of components required; $i = 1, \ldots, I$. An aircraft squadron or detachment deploys with K_i units of each assembly (including installed components plus spares), and has a maintenance shop to perform basic service/repair of components when required. Each item has Markovian failures at rate λ_i, and expected time to repair of ν_i^{-1}. Service times are assumed to be independent and exponentially distributed. The arrival rate of each type of failed item as seen at the maintenance shop will be the individual item failure rate multiplied by the number of items operating at that time. In modeling the aircraft detachment problem, the number of items operating is the number of operational aircraft available. Before considering the aircraft detachment problem directly, a repairman model is considered in which the number of items operating is taken to be the number of items available, i.e., the original population of that item minus the number awaiting repair and or being repaired at that time.

The primary objective of this study is to develop analytic models to analyze the transient behavior of the system based on the effects of a service discipline which is influenced by the numbers of each item awaiting repair. It is especially important in combat support logistics to be able to analyze transient behavior, since, due to changes in combat intensity, a steady-state may never be reached. Besides looking for the mean number of items in the system as a function of time, it is desirable to get a solution for the variances as well. Knowledge of both the mean and variance will allow measures of effectiveness to be calculated which consider, for example, the probability that the number of items available exceeds some threshold. The ultimate application is to assess the adequacy of logistic support on the availability of an operational unit, where that support includes both spares and a single repair facility, e.g., a complex test and repair

stand. Previous work on this problem was done by Latta [Ref. 27], who used simulation to compare several maintenance policies.

In the repair situation, since it is costly in time to switch from job to job before completion, the service disciplines of interest relate to how the next item is selected to commence service, *at the epoch of a previous service completion.* This type of repair service discipline is clearly different from a time-sharing discipline used in some computer systems and communications networks. However, a service discipline approximation used for time-shared systems provides a convenient step towards the analysis of the repair situation.

Processor-sharing is a modeling approximation to the time-sharing discipline. The approach taken in this thesis is to adapt the heavy traffic diffusion analysis of processor-shared systems to study the repair situation in which the next item selected to get service, upon completion of a previous repair, is chosen based on the numbers of each item awaiting service. Specifically, the following is an outline of the development of this chapter:

(1) In Section B., a diffusion approximation is developed for a repairman model in heavy traffic, with processor-sharing, multiple types of queues, and service priority proportional to a function of queue length.

(2) In Section C., a renewal theory approach is used to adapt the model from processor-sharing to the repair situation in which each job is completed before the next job is selected for service.

(3) Sections D. and E. present several numerical examples and an application of the model.

(4) In Section F., the model is extended to general service time distributions.

(5) In Section G., the repairman model is adapted to the aircraft detachment repairman problem.

B. A PROCESSOR-SHARING REPAIRMAN MODEL WITH MULTIPLE TYPES OF JOBS AND PRIORITY SERVICE

Let $N_i(t)$ denote the number of items of type i that are awaiting repair and/or being repaired at time t; collectively denoted by the vector $\mathbf{N}(t) = [N_1(t), N_2(t), \dots, N_I(t)]$.

An item of type i has Markovian failures at rate λ_i, and in this repairman model, the arrival of failed items is proportional to the original population of that item minus the number awaiting repair and/or being repaired at that time. Hence, the time-dependent arrival rate of each type of failed item as seen by the repairman is $\lambda_i(K_i - N_i(t))$, for $i = 1, \ldots, I$. The probability that a failed item of type i arrives at the repair shop in the interval $(t, t + dt)$ is $\lambda_i(K_i - N_i(t))\,dt + o(dt)$.

In processor-sharing, each of the jobs of type i in the system at time t receive a *proportion*, $q_i(\mathbf{N}(t))$, of the processing provided by the server in each interval $(t, t + dt)$. In the traditional processor-sharing model the proportion of service each job of type i receives is identical to the proportion of jobs of that type in the system, i.e., each job present is given equal weight, and $q_i(\mathbf{N}(t))$ is defined by:

$$q_i(\mathbf{N}(t)) = \frac{N_i(t)}{\sum_j N_j(t)} \quad .$$

An equivalent view of processor-sharing is that the server completes infinitesimal time slices (length dt, $dt \to 0$) from a job and then switches to another job (a job of type i) at the end of such a slice with *probability* $q_i(\mathbf{N}(t))$, so only rarely is a job completed and then followed by a jump to a new job. This latter view of $q_i(\mathbf{N}(t))$ as the probability that a job of type i starts (a slice of) service is taken here, so as to set up the processor-sharing model of this section for adaptation to the real repair situation in Section C.

In this processor-sharing model, for a job with an exponentially distributed service time, with mean $1/v_i$, the probability that it completes service in the interval $(t, t + dt)$ is $v_i\, q_i(\mathbf{N}(t))\,dt + o(dt)$. Several papers have reported results using processor-sharing models. See, for example, Coffman, Muntz and Trotter [Ref. 28], Mitra [Ref. 29], and Gaver and Jacobs [Ref. 30].

If the service mechanism is processor-sharing, then $\{\mathbf{N}(t); t \geq 0\}$ is a Markov process in continuous time. If $I = 1$ then this is identical to the classical single-item repairman problem; see Feller [Ref. 31, p. 462] and Gaver and Jacobs [Ref. 30].

Heavy traffic conditions can allow the use of a diffusion approximation to study the time-dependent behavior of the system. Consider the classical single repairman problem with individual machine failure rate λ, service rate v, and K total machines. And let $N(t)$ be the number of machines that have failed and are awaiting repair or being repaired at time t. Iglehart [Ref. 32] has shown that when heavy traffic conditions prevail,

i.e., large K and $\lambda K/v > 1$, $N(t)$ may be approximated by the Ornstein-Uhlenbeck process, hence the diffusion approximation. Several papers have reported models extending diffusion approximations to multivariate birth and death Markov processes, also known as Markov population processes. See for example McNeil and Schach [Ref. 33], Gaver and Lehoczky [Ref. 34], Gaver and Lehoczky [Ref. 35], and Gaver and Jacobs [Ref. 30].

1. Diffusion Approximation

Following the arguments in Gaver and Lehoczky [Ref. 35], the following system of stochastic differential equations are written directly:

$$\begin{aligned} dN_i(t) = {} & \lambda_i (K_i - N_i(t))\, dt - v_i\, q_i(\mathbf{N}(t))\, dt \\ & + \sqrt{\lambda_i (K_i - N_i(t)) + v_i\, q_i(\mathbf{N}(t))}\;\; dW_i(t) \quad , \end{aligned} \tag{5.1}$$

for $i = 1, \ldots, I$; where $\{W_i(t); t \geq 0\}$ are independent standard Wiener processes, i.e., $W_i(0) = 0$; $\{W_i(t); t \geq 0\}$ has stationary and independent increments; and for all $t > 0$ $W_i(t)$ is normally distributed with mean zero and variance t. Here, $N_i(t)$ is a continuous approximation to the actual jump process. The notation $dN_i(t)$ is used to represent the increment $N_i(t + dt) - N_i(t)$. See Karlin and Taylor [Ref. 36] for a systematic development and other examples.

The derivation of Equation (5.1) is as follows. The dt terms represent the infinitesimal drift of $N_i(t)$ from t to $t + dt$, and the $dW_i(t)$ term is the stochastic increment to the process occurring in $(t\,,\; t + dt)$. The form of these terms is obtained from the observation that arrivals and departures act as independent Poisson processes in short time periods. The arrival rate is proportional to the number of remaining items of type i. Departures occur at rate v_i *if* an item of type i is in service at time t, and, under processor-sharing, the probability $q_i(\mathbf{N}(t))$ represents the proportional amount of service that an item of type i receives in the interval $(t\,,\; t + dt)$. Then, since the variance of the Poisson equals the mean, and for large parameter values (large K_i, in this case), the Poisson is approximately Gaussian, this heuristically justifies the coefficient of the Weiner process differential.

Now let $a = \sum_j K_j$, the total population of components, and consider the following normalized process:

$$X_i^a(t) = \frac{N_i(t) - a\, m_i(t)}{\sqrt{a}} \quad ,$$

or

$$N_i(t) = a\, m_i(t) + \sqrt{a}\, X_i^a(t) \quad , \tag{5.2}$$

where $m_i(t)$ are deterministic functions, being approximations to the process means scaled by a; and $X_i^a(t)$ are stochastic elements (random disturbances or *noises* superimposed upon the deterministic approximations to the means). Such a transformation has been used , for example, by McNeil and Schach [Ref. 33], and Gaver and Lehoczky [Ref. 34].

The increment in $N_i(t)$ in $(t\,,\; t+dt)$ is expressed as

$$dN_i(t) = a\, dm_i(t) + \sqrt{a}\, dX_i^a(t) \quad . \tag{5.3}$$

Substituting (5.2) and (5.3) into Equation (5.1) (except, for now, in the $q_i(\mathbf{N}(t))$ term) gives

$$\begin{aligned} a\, dm_i(t) + \sqrt{a}\, dX_i^a(t) = {} & \lambda_i \left(K_i - a\, m_i(t) - \sqrt{a}\, X_i^a(t)\right) dt - \nu_i\, q_i(\mathbf{N}(t))\, dt \\ & + \sqrt{\lambda_i \left(K_i - a\, m_i(t) - \sqrt{a}\, X_i^a(t)\right) + \nu_i\, q_i(\mathbf{N}(t))}\;\; dW_i(t) \quad . \end{aligned} \tag{5.4}$$

The properties of the deterministic and stochastic elements, $m_i(t)$ and $X_i^a(t)$ respectively, as $a \to \infty$ can now be determined. In order to do this, it is necessary to scale the number of components for each item, K_i , expressing these parameters as a fraction of a. Let $K_i = \alpha_i\, a$. Similarly, let the service rate $\nu_i = \mu_i\, a$. Substituting for K_i and ν_i, and dividing through by a, (5.4) becomes

$$\begin{aligned} dm_i(t) + \frac{1}{\sqrt{a}}\, dX_i^a(t) = {} & \lambda_i \left(\alpha_i - m_i(t) - \frac{1}{\sqrt{a}}\, X_i^a(t)\right) dt - \mu_i\, q_i(\mathbf{N}(t))\, dt \\ & + \frac{1}{\sqrt{a}} \sqrt{\lambda_i \left(\alpha_i - m_i(t) - \frac{1}{\sqrt{a}}\, X_i^a(t)\right) + \mu_i\, q_i(\mathbf{N}(t))}\;\; dW_i(t) \quad . \end{aligned} \tag{5.5}$$

The strategy to obtain an analytic solution to these stochastic differential equations is to isolate terms of order 1 and order $1/\sqrt{a}$ yielding a system of ordinary differential equations for the deterministic means, and a system of stochastic differential equations which can be solved to obtain the properties of the noise terms.

It remains to express $q_i(\mathbf{N}(t))$ as a function of $m_i(t)$ and $X_i^a(t)$, when $q_i(\mathbf{N}(t))$ is modeled in a useful and sufficiently smooth (i.e., differentiable) form. For example, if $q_i(\mathbf{N}(t))$ is defined as in the traditional processor-sharing model, then the service priority

rule would be to randomly select the item to receive the next slice of service with probability proportional to the number of that item awaiting service. Thus $q_i(\mathbf{N}(t))$ would be expressed as:

$$q_i(\mathbf{N}(t)) = \frac{N_i(t)}{\sum_j N_j(t)}$$
$$= \frac{a\, m_i(t) + \sqrt{a}\, X_i^a(t)}{\sum_j a\, m_j(t) + \sqrt{a}\, X_j^a(t)} \quad .$$

The strategy for obtaining an appropriate analytic solution only requires terms of order 1 and order $1/\sqrt{a}$. Thus, it is sufficient to derive a first-order asymptotic expansion for $q_i(\mathbf{N}(t))$ in powers of $1/\sqrt{a}$. Rather than limit this development to the expansion of a very specific example of $q_i(\mathbf{N}(t))$, a more general form is considered.

2. Service Priority Proportional to a Smooth Function of Queue Length

A fairly general form for $q_i(\mathbf{N}(t))$ that can be useful for modeling various service policies, and the corresponding expansion to terms of order $1/\sqrt{a}$, is given in the following proposition.

Proposition 5.1: If $q_i(\mathbf{N}(t))$ is of the form

$$q_i(\mathbf{N}(t)) = \frac{w_i\,[b_i + c_i\, N_i(t)]^\gamma}{\sum_j w_j\,[b_j + c_j\, N_j(t)]^\gamma} \quad ,$$

where $w_i \geq 0$, $b_i = a\,\beta_i$, β_i finite, c_i and γ are arbitrary constants, and where $N_i(t)$ is represented by the transformation

$$N_i(t) = a\, m_i(t) + \sqrt{a}\, X_i^a(t) \quad ,$$

then an expansion to terms of order $1/\sqrt{a}$, is given by

$$q_i(\mathbf{N}(t)) = \hat{q}_i(\mathbf{m}(t))\left[\,1 + \frac{1}{\sqrt{a}}\left(X_i^a(t)\,\frac{\gamma\, c_i}{\beta_i + c_i\, m_i(t)} - \sum_j \hat{q}_j(\mathbf{m}(t))\, X_j^a(t)\,\frac{\gamma\, c_j}{\beta_j + c_j\, m_j(t)}\right)\right] \tag{5.6}$$

where

$$\hat{q}_i(\mathbf{m}(t)) = \frac{w_i\,[\beta_i + c_i\,m_i(t)]^\gamma}{\sum_j w_j\,[\beta_j + c_j\,m_j(t)]^\gamma} \quad .$$

Proof: As defined

$$q_i(\mathbf{N}(t)) = \frac{w_i\,[b_i + c_i\,N_i(t)]^\gamma}{\sum_j w_j\,[b_j + c_j\,N_j(t)]^\gamma} \quad . \tag{5.7}$$

Use the representation

$$N_i(t) = a\,m_i(t) + \sqrt{a}\,X_i^a(t) \quad ,$$

and let $b_i = a\,\beta_i$. Then (5.7) becomes

$$\begin{aligned} q_i(\mathbf{N}(t)) &= \frac{w_i\,[a\,\beta_i + c_i\,a\,m_i(t) + c_i\sqrt{a}\;X_i^a(t)]^\gamma}{\sum_j w_j\,[a\,\beta_j + c_j\,a\,m_j(t) + c_j\sqrt{a}\;X_j^a(t)]^\gamma} \\ &= \frac{w_i\,[\beta_i + c_i\,m_i(t) + \frac{1}{\sqrt{a}}\,c_i\,X_i^a(t)]^\gamma}{\sum_j w_j\,[\beta_j + c_j\,m_j(t) + \frac{1}{\sqrt{a}}\,c_j\,X_j^a(t)]^\gamma} \end{aligned} \tag{5.8}$$

Let this last representation be denoted $\hat{q}_i\big(\mathbf{m}(t) + (1/\sqrt{a})\mathbf{X}^a(t)\big)$, where the caret is used to reflect the modification of the original q_i which included the *scaling* of the constant b_i, and the division of numerator and denominator by a^γ. Now, fix t and treat $\hat{q}_i\big(\mathbf{m}(t) + (1/\sqrt{a})\mathbf{X}^a(t)\big)$ as a function of $(1/\sqrt{a})$ alone to obtain an expansion. For brevity, let

$$\phi = \frac{1}{\sqrt{a}} \quad ,$$

$$f_i(\phi) = w_i\,[\beta_i + c_i\,m_i(t) + \phi\,c_i\,X_i^a(t)]^\gamma \quad , \tag{5.9}$$

and

$$g_i(\phi) = \frac{f_i(\phi)}{\sum_j f_j(\phi)} \quad . \tag{5.10}$$

The expansion of $g_i(\phi)$ is

$$g_i(\phi) = g_i(0) + \phi \ \frac{dg_i(0)}{d\phi} + o(\phi) \quad . \tag{5.11}$$

From (5.10)

$$\begin{aligned} \frac{dg_i(\phi)}{d\phi} &= \frac{1}{\sum_j f_j(\phi)} \ \frac{df_i(\phi)}{d\phi} - \frac{f_i(\phi)}{\left(\sum_j f_j(\phi)\right)^2} \ \frac{d}{d\phi} \left(\sum_j f_j(\phi)\right) \\ &= \frac{1}{\sum_k f_k(\phi)} \ \frac{df_i(\phi)}{d\phi} - g_i(\phi) \sum_j \frac{1}{\sum_k f_k(\phi)} \ \frac{df_j(\phi)}{d\phi} \quad . \end{aligned} \tag{5.12}$$

From (5.9)

$$\begin{aligned} \frac{df_i(\phi)}{d\phi} &= w_i[\beta_i + c_i\, m_i(t) + \phi\ c_i\, X_i^a(t)]^{\gamma-1} (\gamma)\ (c_i\, X_i^a(t)) \\ &= w_i[\beta_i + c_i\, m_i(t) + \phi\ c_i\, X_i^a(t)]^{\gamma} \ \frac{\gamma\ c_i\, X_i^a(t)}{\beta_i + c_i\, m_i(t) + \phi\ c_i\, X_i^a(t)} \\ &= f_i(\phi) \ \frac{\gamma\ c_i\, X_i^a(t)}{\beta_i + c_i\, m_i(t) + \phi\ c_i\, X_i^a(t)} \quad . \end{aligned} \tag{5.13}$$

Substituting (5.13) into (5.12)

$$\begin{aligned} \frac{dg_i(\phi)}{d\phi} &= \frac{f_i(\phi)}{\sum_k f_k(\phi)} \ \frac{\gamma\ c_i\, X_i^a(t)}{\beta_i + c_i\, m_i(t) + \phi\ c_i\, X_i^a(t)} - g_i(\phi) \sum_j \frac{f_j(\phi)}{\sum_k f_k(\phi)} \ \frac{\gamma\ c_j\, X_j^a(t)}{\beta_j + c_j\, m_j(t) + \phi\ c_j\, X_j^a(t)} \\ &= g_i(\phi) \ \frac{\gamma\ c_i\, X_i^a(t)}{\beta_i + c_i\, m_i(t) + \phi\ c_i\, X_i^a(t)} - g_i(\phi) \sum_j g_j(\phi) \ \frac{\gamma\ c_j\, X_j^a(t)}{\beta_j + c_j\, m_j(t) + \phi\ c_j\, X_j^a(t)} \\ &= g_i(\phi) \left(\frac{\gamma\ c_i\, X_i^a(t)}{\beta_i + c_i\, m_i(t) + \phi\ c_i\, X_i^a(t)} - \sum_j g_j(\phi) \ \frac{\gamma\ c_j\, X_j^a(t)}{\beta_j + c_j\, m_j(t) + \phi\ c_j\, X_j^a(t)} \right) \quad . \end{aligned} \tag{5.14}$$

Evaluating (5.14) at $\phi = 0$

$$\frac{dg_i(0)}{d\phi} = g_i(0)\left(\frac{\gamma\, c_i\, X_i^a(t)}{\beta_i + c_i\, m_i(t)} - \sum_j g_j(0)\, \frac{\gamma\, c_j\, X_j^a(t)}{\beta_j + c_j\, m_j(t)} \right) \quad . \tag{5.15}$$

Substituting (5.15) into (5.11)

$$g_i(\phi) = g_i(0)\left[1 + \phi \left(\frac{\gamma\, c_i\, X_i^a(t)}{\beta_i + c_i\, m_i(t)} - \sum_j g_j(0)\, \frac{\gamma\, c_j\, X_j^a(t)}{\beta_j + c_j\, m_j(t)} \right) \right] + o(\phi) \quad . \tag{5.16}$$

Recalling that $\phi = (1/\sqrt{a}\,)$, and

$$\begin{aligned} g_i(\phi) &= \frac{w_i\,[\beta_i + c_i\, m_i(t) + \frac{1}{\sqrt{a}}\, c_i\, X_i^a(t)]^\gamma}{\sum_j w_j\,[\beta_j + c_j\, m_j(t) + \frac{1}{\sqrt{a}}\, c_j\, X_j^a(t)]^\gamma} \\ &\equiv \hat{q}_i\left(\mathbf{m}(t) + \frac{1}{\sqrt{a}}\, \mathbf{X}(t) \right) \quad . \end{aligned} \tag{5.17}$$

Then

$$\begin{aligned} g_i(0) &= \frac{w_i\,[\beta_i + c_i\, m_i(t)]^\gamma}{\sum_j w_j\,[\beta_j + c_j\, m_j(t)]^\gamma} \\ &\equiv \hat{q}_i(\mathbf{m}(t)) \quad , \end{aligned} \tag{5.18}$$

and the result is obtained. ■

Specializations of the general form of $q_i(\mathrm{N}(t))$ given in Proposition 5.1 will be introduced in Section C.

3. Diffusion Approximation (continued)

Returning now to the stochastic differential equations, substituting (5.6) into (5.5) and isolating terms of order 1 and $1/\sqrt{a}$, the following sets of equations are obtained.

Equations of Order 1. The equations of order 1 form the following system of ordinary differential equations:

$$dm_i(t) = \lambda_i (\alpha_i - m_i(t))\, dt - \mu_i\, \hat{q}_i(\mathbf{m}(t))\, dt \quad ; \tag{5.19}$$

for $i = 1, \ldots, I$. With given initial conditions, a solution can be obtained by numerical methods, which provides a deterministic approximation to the *scaled* mean queue lengths as a function of time.

Equations of Order $1/\sqrt{a}$. The equations of order $1/\sqrt{a}$ form the following system of *stochastic* differential equations:

$$\begin{aligned} dX_i^a(t) = & -\lambda_i X_i^a(t)\, dt - \mu_i \frac{\gamma\, c_i}{\beta_i + c_i\, m_i(t)}\, \hat{q}_i(\mathbf{m}(t))\,(1 - \hat{q}_i(\mathbf{m}(t)))\, X_i^a(t)\, dt \\ & + \mu_i \sum_{j \neq i} \frac{\gamma\, c_j}{\beta_j + c_j\, m_j(t)}\, \hat{q}_i(\mathbf{m}(t))\, \hat{q}_j(\mathbf{m}(t))\, X_j^a(t)\, dt \\ & + \sqrt{\lambda_i (\alpha_i - m_i(t)) + \mu_i\, \hat{q}_i(\mathbf{m}(t))}\, dW_i(t) \quad ; \end{aligned} \tag{5.20}$$

for $i = 1, \ldots, I$.

It is noted that the effect of boundaries on the evolution of the system has not been included. In the deterministic equations (5.19), inclusion would constrain all $m_i(t)$ and their sum to be within [0, 1]. A *heavy traffic condition* will imply that, with high probability, the system will evolve away from the boundary at zero and return very rarely. Such a condition may be derived from (5.19). As $m_i(t) \to 0$ its derivative must become strictly positive to move $m_i(t)$ away from zero. Thus

$$\lambda_i\, \alpha_i - \mu_i\, \hat{q}_i(0^+) > 0 \quad , \tag{5.21}$$

or

$$\frac{\lambda_i\, \alpha_i}{\mu_i} > \hat{q}_i(0^+) \quad ; \tag{5.21}$$

for $i = 1, \ldots, I$. Since, $0 \leq \hat{q}_i(0^+) \leq 1$, for all i, the following *sufficient* heavy traffic condition, hereafter HTC, is suggested:

$$\frac{\lambda_i\, \alpha_i}{\mu_i} > 1 \quad ; \tag{5.22}$$

for $i = 1, \ldots, I$. HTC is clearly stronger than required. Since the $\hat{q}_i(0^+)$ sum to 1, it should not be necessary for (5.22) to be satisfied simultaneously for all i. The *necessary* heavy traffic condition may be obtained by summing (5.21) over all i. This gives

$$\sum_i \frac{\lambda_i \alpha_i}{\mu_i} > \sum_i \hat{q}_i(0^+) \quad ,$$

or

$$\sum_i \frac{\lambda_i \alpha_i}{\mu_i} > 1 \quad . \tag{5.23}$$

This will be referred to as an aggregated heavy traffic condition (AHTC).

The upper boundary is implicitly enforced by the $(\alpha_i - m_i(t))$ coefficient of the arrival rates. The stochastic equation (5.20) are only valid for

$$-\sqrt{a}\, m_i(t) \leq X_i^a(t) \leq \sqrt{a}\,(\,\alpha_i - m_i(t)\,) \quad ,$$

so that for $0 < m_i(t) < \alpha_i$, the boundary can be ignored as $a \rightarrow \infty$. For these models, it is assumed that the AHTC holds.

4. Solution of the Stochastic Differential Equations

To write (5.20) in matrix form, let

$$\mathbf{X}^a(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_I(t) \end{bmatrix} \quad ,$$

and

$$\mathbf{W}(t) = \begin{bmatrix} W_1(t) \\ W_2(t) \\ \vdots \\ W_I(t) \end{bmatrix} \quad .$$

Then (5.20) becomes

$$d\mathbf{X}^a(t) = \mathbf{H}(t)\,\mathbf{X}^a(t)\,dt + \mathbf{B}(t)\,d\mathbf{W}(t) \quad ; \tag{5.24}$$

where $\mathbf{H}$ is an $I \times I$ matrix with elements

$$H_{ii}(t) = -\lambda_i - \mu_i \frac{\gamma c_i}{\beta_i + c_i m_i(t)} \hat{q}_i(\mathbf{m}(t))\,(1 - \hat{q}_i(\mathbf{m}(t))) \quad ,$$

and

$$H_{ij}(t) = \mu_i \frac{\gamma c_j}{\beta_j + c_j m_j(t)} \hat{q}_i(\mathbf{m}(t))\,\hat{q}_j(\mathbf{m}(t)) \quad ,$$

for $i \neq j$; $\mathbf{B}(t)$ is an $I \times I$ diagonal matrix with elements

$$B_{ii}(t) = \sqrt{\lambda_i\,(\alpha_i - m_i(t)) + \mu_i\,\hat{q}_i(\mathbf{m}(t))} \quad ;$$

and with initial conditions, $\mathbf{X}^a(0) = \mathbf{0}$.

If $m_i(t)$ satisfies (5.19) then the results of Kurtz [Ref. 37] and Barbour [Ref. 38] imply as $a \to \infty$ that $\{\mathbf{X}^a(t);\, t \geq 0\}$ will converge weakly to $\{\mathbf{X}(t);\, t \geq 0\}$, governed by the stochastic differential equation

$$d\mathbf{X}(t) = \mathbf{H}(t)\,\mathbf{X}(t)\,dt + \mathbf{B}(t)\,d\mathbf{W}(t) \quad . \tag{5.25}$$

Gaver and Jacobs [Ref. 30, Appendix] outline the mathematical foundation upon which the diffusion approximation of this chapter may be rigorously based.

Equation (5.25) characterizes an Ornstein-Uhlenbeck process, for which several results are given by Arnold [Ref. 39, p.143]. Specifically, $\mathbf{X}(t)$ has a multivariate normal distribution with mean $\mathbf{0}$ and variance-covariance matrix $\mathbf{V}(t)$ which satisfies the following system of ordinary differential equations:

$$\frac{d\mathbf{V}(t)}{dt} = \mathbf{H}(t)\,\mathbf{V}(t) + \mathbf{V}(t)\,\mathbf{H}'(t) + \mathbf{B}(t)\,\mathbf{B}'(t) \quad . \tag{5.26}$$

Recalling that

$$\mathbf{N}(t) = a\,\mathbf{m}(t) + \sqrt{a}\,\mathbf{X}^a(t) \quad ,$$

the following result has been obtained:

Result 5.1: Under heavy traffic conditions ($\sum_i \lambda_i K_i / v_i > 1$), for a large system ($a \to \infty$, $a = \sum_i K_i$, where all $K_i \to \infty$ simultaneously and in fixed proportion), $\mathbf{N}(t)$ is multivariate normal (Gaussian) with mean $a\,\mathbf{m}(t)$ and variance-covariance matrix $a\,\mathbf{V}(t)$.

From this result it is possible to obtain estimates of the mean and variance of time-dependent queue lengths that result from adoption of a particular service policy modeled by the function $q_i(\mathbf{N}(t))$.

Solutions to the systems of ordinary differential equations (5.19) and (5.25) can be obtained by straightforward numerical methods. Writing

$$\mathbf{V}(t) = \begin{bmatrix} \sigma_{11}(t) & \sigma_{12}(t) & \cdots & \sigma_{1I}(t) \\ \sigma_{21}(t) & \sigma_{22}(t) & \cdots & \sigma_{2I}(t) \\ & \vdots & & \\ \sigma_{I1}(t) & \sigma_{I2}(t) & \cdots & \sigma_{II}(t) \end{bmatrix} ,$$

and making the required substitutions and multiplications, differential equations for the elements of $\mathbf{V}(t)$ are

$$\frac{d\sigma_{ii}(t)}{dt} = (B_{ii}(t))^2 + 2\sum_{j=1}^{I}(H_{ij}(t)\,\sigma_{ij}(t)) \quad ; \tag{5.27}$$

and

$$\frac{d\sigma_{ij}(t)}{dt} = \sum_{k=1}^{I}[(H_{ik}(t)\,\sigma_{jk}(t)) + (H_{jk}(t)\,\sigma_{ik}(t))] \quad ; \tag{5.28}$$

for $i \neq j$.

C. DYNAMIC-SERVICE-SELECTION: THE NATURAL ALTERNATIVE TO PROCESSOR-SHARING

An alternative to the processor-sharing model represented by Equation (5.1) can be developed with a renewal theory approach to the service completion process. This alternative model reflects that the service to another item can only commence when a previous repair is completed, where the item to receive the next service is selected probabilistically. This discipline will be referred to as *dynamic-service-selection* (DSS).

As in the previous processor-sharing model, the functional form of the probabilities used to select the next item for service is chosen so as to model priority for service as a function of queue length. The general form for $q_i(\mathbf{N}(t))$ introduced in Proposition 5.1 may be used in this repair situation, but with an important conceptual distinction. In

dynamic-service-selection, $q_i(\mathbf{N}(t))$ can no longer be interpreted as a *proportion* of service received by an item of type i in the interval $(t, t+dt)$, as was the case in processor sharing. Now, $q_i(\mathbf{N}(t))$ is only defined for t marking the epoch of a service completion, and has only one interpretation -- the probability that an item of type i is selected to receive the next available service.

1. Renewal Theory Approach to the Repair Service Situation

Consider a random variable which is the number of service completions of type i in the interval $[0, t]$, and let C_i denote a cycle length between successive service completions of type i. A renewal theory result states that if C_i has expectation $E[C_i]$, and variance $\text{var}[C_i]$, then the number of service completions in time t has expectation $t\,E[C_i]^{-1}$, and variance $t\,\text{var}[C_i]\,E[C_i]^{-3}$ asymptotically as $t \to \infty$; see Feller [Ref. 40, p. 372].

An alternative to the processor-sharing model represented by Equation (5.1) can now be written directly by replacing the $v_i\,q_i(\mathbf{N}(t))$ terms which were justified by the assumption that service completions resemble a Poisson process in short time periods. Using the mean and variance coefficients obtained from the renewal theory approach, Equation (5.1) becomes:

$$\begin{aligned} dX_i(t) = {} & \lambda_i (K_i - X_i(t))\,dt \; - \; E[C_i]^{-1}\,dt \\ & + \sqrt{\lambda_i (K_i - X_i(t)) + \text{var}[C_i]\,E[C_i]^{-3}} \quad dW_i(t) \quad , \end{aligned} \tag{5.29}$$

for $i = 1, \ldots, I$; where $\{W_i(t); t \geq 0\}$ are independent standard Wiener processes.

To obtain closed-form approximations for $E[C_i]$ and $\text{var}[C_i]$, consider the length of a cycle when the system is in state $\mathbf{n} = [n_1, n_2, \ldots, n_I]$, where $n_i = X_i(t)$. The cycle time C_i begins when a type i item completes service and ends when the next type i item completes service. The cycle time C_i may be written as:

$$C_i = \begin{cases} S_i & \text{with probability } q_i(\mathbf{n}) \\ S_j + C_i^{*} & \text{with probability } q_j(\mathbf{n}),\ j \neq i \ ; \end{cases} \tag{5.30}$$

where S_i is the service time of item i. The justification for Equation (5.30) is that when an item of type i completes service, either another item of type i is chosen to start service, an event of probability $q_i(\mathbf{n})$, or an item of type j, $j \neq i$, is chosen and after it completes service the cycle starts over in a state near enough to $\mathbf{n}$ so that C_i^{*} has the same distribution as C_i, an event of probability $q_j(\mathbf{n})$, $j \neq i$. This is justified under the

heavy traffic and large a conditions previously specified for the use of the diffusion approximation. Taking expectations, and using $E[S_i] = \frac{1}{v_i}$, gives

$$E[C_i] = \sum_j \frac{1}{v_j} q_j(\mathbf{n}) \;+\; E[C_i]\,(1 - q_i(\mathbf{n})) \;\; .$$

This is solved for $E[C_i]$, giving

$$E[C_i] = \frac{1}{q_i(\mathbf{n})} \sum_j \frac{1}{v_j} q_j(\mathbf{n}) \;\; . \tag{5.31}$$

This approach leads to a simple alteration of the differential equations for the mean queue lengths, $m_i(t)$, derived under processor-sharing, that enables them to describe DSS.

Proposition 5.2: If $q_i(\mathbf{n})$ is of the form

$$q_i(\mathbf{n}) = \frac{w_i f_i(n_i)}{\sum_k w_k f_k(n_k)} \;\; ,$$

where $f_i(n_i)$ is an arbitrary function of n_i, and $\tilde{w}_i$ and $\tilde{q}_i(\mathbf{n})$ are defined by $\tilde{w}_i = \frac{w_i}{v_i}$, and

$$\tilde{q}_i(\mathbf{n}) = \frac{\tilde{w}_i f_i(n_i)}{\sum_k \tilde{w}_k f_k(n_k)} \;\; , \tag{5.32}$$

then

$$E[C_i]^{-1} = v_i\, \tilde{q}_i(\mathbf{n}) \;\; . \tag{5.33}$$

Proof: From Equation (5.31):

$$E[C_l]^{-1} = \frac{q_l(\mathbf{n})}{\sum_j \frac{1}{v_j}\, q_j(\mathbf{n})}$$

$$= v_l \left[\frac{\frac{1}{v_l}\, q_l(\mathbf{n})}{\sum_j \frac{1}{v_j}\, q_j(\mathbf{n})} \right] .$$

Substituting for $q_l(\mathbf{n})$ and simplifying:

$$E[C_l]^{-1} = v_l \left[\frac{\frac{w_l}{v_l}\, f_l(n_l)}{\sum_j \frac{w_j}{v_j}\, f_j(n_j)} \right] .$$

Letting $\tilde{w}_i = \frac{w_i}{v_i}$:

$$E[C_l]^{-1} = v_l \left[\frac{\tilde{w}_l f_l(n_l)}{\sum_j \tilde{w}_j f_j(n_j)} \right] .$$

Defining

$$\tilde{q}_l(\mathbf{n}) = \frac{\tilde{w}_l f_l(n_l)}{\sum_k \tilde{w}_k f_k(n_k)} ,$$

then

$$E[C_l]^{-1} = v_l\, \tilde{q}_l(\mathbf{n}) \ . \ \blacksquare$$

In this form it is seen that by using modified weights, i.e., dividing the original item weights by the item service rates, the differential equations for the means are of the same form as in the processor-sharing model. These modified weights reduce to the original weights in the special case when service rates are equal. Thus the modified weights can be interpreted as the correction to the means to modify the service discipline from processor-sharing to dynamic-service-selection.

The form of $q_i(\mathbf{n})$ in Proposition 5.2 is more general than the form given in Proposition 5.1 so this simple modification applies to specializations of Proposition 5.1. For example, if $q_i(\mathbf{n})$ is defined by the following:

$$q_i(\mathbf{n}) = \frac{w_i\,(n_i)^p}{\sum_j w_j\,(n_j)^p} \quad ,$$

Then

$$\tilde{q}_i(\mathbf{n}) = \frac{\tilde{w}_i\,(n_i)^p}{\sum_j \tilde{w}_j\,(n_j)^p} \quad ;$$

where $\tilde{w}_i = \frac{w_i}{v_i}$. For $p = 1$, and all $w_i = 1$, this modification is equivalent to the processor-sharing approximation of FCFS reported by Gaver and Lehoczky [Ref. 41]. This FCFS approximation is discussed in a following section.

Starting again with (5.30), a similar derivation gives the second moment and hence variance of C_i:

$$\mathrm{var}[C_i] = \frac{2}{q_i(\mathbf{n})} \sum_j \frac{q_j(\mathbf{n})}{v_j^2} + \mathrm{E}[C_i]^2 - \frac{2}{v_i}\,\mathrm{E}[C_i] \quad . \tag{5.34}$$

Using (5.33), (5.32), and (5.34), the following term, which is the approximate component of the variance in $dX_i(t)$ due to departures, is derived:

$$\frac{\mathrm{var}[C_i]}{\mathrm{E}[C_i]^3} = v_i\,\tilde{q}_i(\mathbf{n}) \left(1 + 2\,\tilde{q}_i(\mathbf{n}) \left\{ v_i \sum_j \frac{\tilde{q}_j(\mathbf{n})}{v_j} - 1 \right\} \right) \quad . \tag{5.35}$$

In the special case of equal service rates, (5.35) reduces to $v_i\,\tilde{q}_i(\mathbf{n}) = v_i\,q_i(\mathbf{n})$, which is the same as in the processor-sharing model. Thus the term within the large parentheses can be interpreted as the variance correction to modify the service discipline from processor-sharing to dynamic-service-selection.

Using (5.33) and (5.35), the model given by (5.29) becomes

$$dN_i(t) = \lambda_i (K_i - N_i(t))\, dt \; - \; v_i\, \tilde{q}_i(\mathbf{n})\, dt$$
$$+ \sqrt{\lambda_i (K_i - N_i(t)) + v_i\, \tilde{q}_i(\mathbf{n}) \left(1 + 2\, \tilde{q}_i(\mathbf{n}) \left\{ v_i \sum_j \frac{\tilde{q}_j(\mathbf{n})}{v_j} \; - \; 1 \right\} \right)} \; dW_i(t)\,, \tag{5.36}$$

for $i = 1, \ldots, I$. Applying the diffusion approximation to (5.36), the following differential equations are obtained for the scaled mean queue lengths and covariance matrix elements:

$$dm_i(t) = \lambda_i (\alpha_i - m_i(t))\, dt - \mu_i\, \tilde{q}_i(\mathbf{m}(t))\, dt \quad ; \tag{5.37}$$

for $i = 1, \ldots, I$;

$$\frac{d\sigma_{ii}(t)}{dt} = (B_{ii}(t))^2 + 2 \sum_{j=1}^{I} (H_{ij}(t)\, \sigma_{ij}(t)) \quad ; \tag{5.38}$$

and

$$\frac{d\sigma_{ij}(t)}{dt} = \sum_{k=1}^{I} [(H_{ik}(t)\, \sigma_{jk}(t)) + (H_{jk}(t)\, \sigma_{ik}(t))] \quad ; \tag{5.39}$$

for $i \neq j$; where

$$H_{ii}(t) = -\lambda_i - \mu_i \frac{\gamma\, c_i}{\beta_i + c_i\, m_i(t)}\, \tilde{q}_i(\mathbf{m}(t))\, (1 - \tilde{q}_i(\mathbf{m}(t))) \quad ,$$

$$H_{ij}(t) = \mu_i \frac{\gamma\, c_j}{\beta_j + c_j\, m_j(t)}\, \tilde{q}_i(\mathbf{m}(t))\, \tilde{q}_j(\mathbf{m}(t)) \quad ,$$

for $i \neq j$,

$$B_{ii}(t)^2 = \lambda_i (\alpha_i - m_i(t)) + V_i \quad ,$$

$$V_i = \mu_i\, \tilde{q}_i(\mathbf{m}(t)) \left(1 + 2\, \tilde{q}_i(\mathbf{m}(t)) \left\{ \mu_i \sum_j \frac{\tilde{q}_j(\mathbf{m}(t))}{\mu_j} \; - \; 1 \right\} \right) \quad .$$

and where

$$\widetilde{q}_i(\mathbf{m}(t)) = \frac{\frac{w_i}{\mu_i}\,[\beta_i + c_i\, m_i(t)]^{\gamma}}{\sum_j \frac{w_j}{\mu_j}\,[\beta_j + c_j\, m_j(t)]^{\gamma}} \quad .$$

The next sections introduce reasonable specific forms for the service selection probabilities, $q_i(\mathrm{N}(t))$.

2. Probabilistic-Longest-Line Service Discipline

The general form of $q_i(\mathrm{N}(t))$ given in Proposition 5.1 may be specialized to a family of functions that can be useful for analysis of a rule that gives service priority based on queue length. If b_i is set equal to 0, c_i set to 1, and γ set to p, then the general function becomes

$$f_i(N_i(t)) = w_i\, N_i(t)^p \quad .$$

This gives the service probability

$$q_i(\mathrm{N}(t)) = \frac{w_i\, N_i(t)^p}{\sum_j w_j\, N_j(t)^p} \quad . \tag{5.40}$$

This family of functions will be collectively referred to as a model of the Probabilistic-Longest-Line with parameter p (PLL;p) service discipline.

Here w_i is a *weight* for items of type i. This weight could be a function of the failure rate or average service time for an item of that type, or it could be a reasonable measure of the mission importance of an item of that type. Alternatively, w_i can be regarded as a decision variable at the disposal of an executive who wishes to optimize some feature of the combined backlog.

For $p = 1$ and $w_i = 1$ for $i = 1, \ldots, I$, $q_i(\mathrm{N}(t))$ represents the traditional processor-sharing discipline.

Higher values for p could be used to get an analytical solution which approximates a rule which selects the item with the longest queue for service, which will be referred to as the *Longest-Line-First* (LLF) discipline.

3. First-Come-First-Served Service Discipline

Gaver and Lehoczky [Ref. 41] demonstrated that, in the special case corresponding to what is here called PLL;1, if $w_i = 1/\mu_i$, then (5.40) will lead to the system reaching approximately the same steady-state as with the *first-come-first-served* (FCFS) service discipline. They stated, however, that the transient behavior had not been validated. With the use of this diffusion model, the accuracy, during the transient response of the system, of approximating FCFS by PLL;1 with $w_i = 1/\mu_i$, is now confirmed numerically. This allows a computationally feasible analytic study of FCFS which by Markov chain methods would require a significantly expanded state space. Selected numerical results will be given later.

4. Probabilistic-Lowest-Availability Service Discipline

The general form of $q_i(\mathrm{N}(t))$ given in Proposition 5.1 may also be specialized to a family of functions that can be useful for analysis of a rule that gives service priority based on item availability. If b_i is set equal to K_i, c_i set to -1, and γ set to $-p$, for positive p, then the general function becomes

$$q_i(\mathrm{N}(t)) = \frac{w_i\,(K_i - N_i(t))^{-p}}{\sum_j w_j\,(K_j - N_j(t))^{-p}} \quad . \tag{5.41}$$

This family of functions will be collectively referred to as a model of the Probabilistic-Lowest-Availability with parameter p (PLA;p) service discipline.

Here w_i is a weight as discussed in the PLL;p service discipline. The difference $(K_i - N_i(t))$ is the item availability. For $p = 1$, $q_i(\mathrm{N}(t))$ represents probabilistic service proportional to the weighted inverse of item availability (i.e., a lower number of available items implies higher priority for service).

Higher values for p can be used to get an analytical solution which approximates a rule which selects the item with the lowest availability for service, which will be referred to as the *Lowest-Availability-First* (LAF) discipline.

5. Morrison's Generating Function Approach

Another approach to obtaining a steady-state solution for this problem has been proposed by Morrison; see Morrison, Gaver, and Pilnick [Ref. 42]. Working in the original discrete state space of the continuous time Markov process, $P_t(\mathbf{n};t)$, where

$\mathbf{n} = (n_1, \dots, n_I)$, is defined as the probability that there are n_j items of type j, $j = 1, \dots, I$, in the system at time t, and that an item of type i is in service. Suitable boundary conditions are also defined. Transition probabilities are then used to write down the Kolmogorov Forward Equations for the system. Then, defining the limiting probabilities

$$\mathrm{p}_i(\mathbf{n}) = \lim_{t \to \infty} \mathrm{P}_i(\mathbf{n};t) \quad ,$$

steady-state balance equations are derived. The strategy at this point is to work with a transform, or generating function, of the limiting probabilities defined as

$$u_i(\mathbf{x}) \;=\; \sum_{0 \le \mathbf{n} \le \mathbf{K}} \mathrm{p}_i(\mathbf{n})\, x_1^{n_1} \;\dots\; x_I^{n_I} \quad ,$$

and the partial derivatives

$$x_j \frac{\partial u_i(\mathbf{x})}{\partial x_j} \;=\; \sum_{0 \le \mathbf{n} \le \mathbf{K}} n_j\, \mathrm{p}_i(\mathbf{n})\, x_1^{n_1} \;\dots\; x_I^{n_I} \quad .$$

Anticipating these transformations, the balance equations are summed over the states and multiplied by the appropriate products of the transform arguments, x_i. Then, the parameters and variables of the problem are scaled as in the diffusion approximation, i.e., $K_i = a\,\alpha_i$, $v_i = a\,\mu_i$ and now $x_i = 1 - (\xi_i/a)$, introducing $\psi(\underset{\sim}{\xi}) = u_i(\mathbf{x})$. An asymptotic expansion of $\psi(\underset{\sim}{\xi})$ is assumed. The lowest order terms then lead to a system of partial differential equations for which the form of the solution can be recognized, which ultimately leads to the steady-state solution for the mean numbers in the system. The next higher order terms of the expansion lead to the steady-state solution for the covariances of the numbers in the system in the special case where the arrival rates are equal and the service discipline is modeled by

$$q_i(\mathbf{m}(t)) = \frac{\frac{1}{\mu_i}\, m_i(t)}{\sum_j \frac{1}{\mu_j}\, m_j(t)} \quad ;$$

for $i = 1, \dots, I$.

The steady-state solution by this method, and by the diffusion approximation agree in the means, but not exactly in the covariances for different v_i. There is agreement when the service rates are equal. For all examples treated, the agreement has been usefully good, even when service rates differ. Appendix F has the resulting steady-state expressions by both methods. In the special cases where Morrison's solution is applicable, the numerical examples which follow compare the results. The principal advantage of the diffusion approximation over this approach is that this method does not easily give the transient response of the system. Also, the above method cannot provide information when services are not Markovian, whereas the diffusion approximation does an adequate job. (The extension of the diffusion approximation DSS model to service times with a general distribution is taken up in a later section.) At present, the preceding diffusion approximation provides the only analytical-numerical approach to the service problem described that can be used for time-dependent logistics applications.

D. NUMERICAL EXAMPLES

Several example problems were run to examine the results of using the diffusion approximation. The numerical solution results were compared to corresponding simulation results, and in very special cases, to direct analytic results.

All numerical solutions of the differential equations were carried out on an IBM 3033 computer at the Naval Postgraduate School using the IMSL Release 10 subroutine IVPAG with the Adams-Moulton method (see IMSL [Ref. 43]). Solutions for the mean and variance of queue lengths were computed for cases in which repair service is provided probabilistically using functions of the form considered in Proposition 5.1.

All simulations were also carried out on the IBM 3033 computer at the Naval Postgraduate School, using the LLRANDOMII random number generating package (see Lewis and Uribe [Ref. 44]). Time-dependent queue lengths were simulated. An event clock was advanced at either job arrivals or service completions, at which time the queue lengths were either incremented or decremented accordingly. The current queue lengths were recorded at fixed discrete time steps as the process evolved. For each case, 500 independent replications were completed. Sample moments at each integer time unit were computed, for comparison with the results of the numerical solution of the differential equations obtained from the diffusion approximation. In addition, sample data were taken from the simulation to assess the validity of the assumption of normality underlying the heavy traffic model. In different cases in the simulations, the item to re-

ceive service following a service completion was either selected probabilistically using probabilities corresponding to the diffusion approximation cases, or *deterministically* from a distinguished queue, such as the longest (i.e., using the longest-line-first (LLF) discipline).

In addition to the time-dependent results from the diffusion approximation and the simulation, steady-state moments, using both the diffusion approximation and Morrison's results, were computed in some cases to check the time-dependent results. The latter should agree with the former as time increases. A simple check was to consider the special case in which there was only one type of item so that the problem reduced to the classical repairman problem for which analytical steady-state mean and variance could be directly computed.

Another check was to use the diffusion approximation to directly compute steady-state mean queue lengths to check the results obtained from the numerical solution of the differential equations. The method used to compute the steady-state mean involved setting the rate of change in the deterministic differential equations (5.19) to zero, summing over all item types, using Newton's method to find the fixed point for the denominator of the $q_i(\mathbf{m}(t))$ terms, then backsolving for each steady-state $m_i(t)$; see Morrison, Gaver, and Pilnick [Ref. 42]. Details are found in Appendix F. Similarly, the diffusion approximation was used to directly compute the steady-state queue length variances in the special case of equal failure rates.

It may be mentioned here, that on the mainframe computer, the diffusion approximation approach took only a few seconds to return a numerical solution to the differential equations in the longest cases. The program was written in FORTRAN and could be compiled and run on a personal computer. An implementation of the diffusion approximation approach on a PC would provide a maintenance policy decision maker with a tool to reasonably compare alternative policies. In contrast to the rapid computation of the diffusion approximation solution, the simulation took approximately fifteen minutes to run on the mainframe, and would run much longer on a PC.

Example 5.1: As an example of an analysis of a repair policy that gives service priority based on which queue is longest, i.e., PLL;p service, numerical examples with a common input and various solution methods are compared. The inputs for this example are shown in Figure 12. This example is a special case in which all service rates are equal so that the system behaves as if the service discipline were processor-sharing.

i	1	2	3	4	5
K_i	100.	110.	120.	130.	140.
λ_i	0.011	0.012	0.013	0.014	0.015
v_j	3.0	3.0	3.0	3.0	3.0
w_i	1.0	1.0	1.0	1.0	1.0
$N(0)$	0	0	0	0	0

Figure 12. Example 5.1 Inputs

Results for Example 5.1 are obtained and presented for each of the following cases.

a. Case APLL;1 (Diffusion Approximation, **PLL;1** Service). This case is the numerical solution obtained from the diffusion approximation in which the service rule is modeled by the probabilistic form

$$q_i(\mathbf{N}(t)) = \frac{w_i \, N_i(t)^p}{\sum_j w_j \, N_j(t)^p} \quad ; \tag{5.42}$$

with the parameter p set equal to 1, which approximates FCFS.

b. Case APLL;p (Diffusion Approximation, **PLL;p** Service). This case is the is the same as Case APLL;1, but with the parameter p set equal to a high value, in this example 2, 10, 20, and finally as high as 30, to get an analytical solution which approximates deterministic service of the longest queue.

c. Case SPLL;1 (Simulation, **PLL;1** Service). This case is the simulation outcome in which the service discipline is randomized selection of the next queue for service, upon each service completion, in accordance with probabilities using (5.42).

d. Case SFCFS (Simulation, **FCFS** Service). This case is the simulation outcome in which the service discipline is first-come-first-serve.

e. Case SLLF (Simulation, **LLF** Service). This case is the simulation outcome in which the service discipline is to serve the longest queue upon each service completion.

For Example 5.1, a typical resulting queue length as a function of time is shown in Figure 13. Results are shown for the queue developed for one of the items, for the sol-

utions of Cases APLL;1 and SPLL;1, to compare diffusion approximation results with corresponding simulation results.

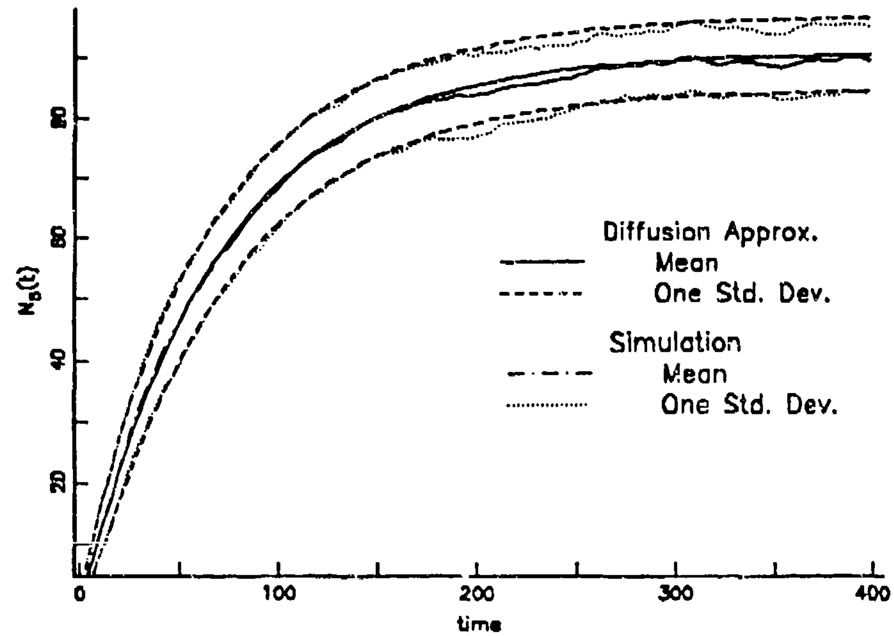


Figure 13. Example 5.1 Queue Length vs. Time

Numerical results for Example 5.1 are summarized in Figure 14. Results are given for each case at time increments of 100 time units. The values listed are the mean queue lengths with standard deviations in parentheses. Standard errors and confidence intervals for the point estimates for the means obtained from the simulation are omitted from the tabulated results to avoid more clutter in the table. Upper and lower .95 confidence limits for the means are the point estimate $\pm .0877$ times the corresponding estimate for the standard deviation (i.e., about $\pm$ 10% of the standard deviation). Upper and lower .95 confidence limits for the standard deviations are .942 and 1.066 times the point estimate (i.e., about $\pm$ 5%); see Lewis and Orav [Ref. 45].

Case	$N_1(t)$	$N_2(t)$	$N_3(t)$	$N_4(t)$	$N_5(t)$
t=100					
SFCFS	40.5 (5.3)	47.4 (5.5)	53.8 (6.3)	61.2 (6.6)	68.4 (6.8)
SPLL; 1	40.0 (5.6)	47.2 (5.6)	53.6 (5.9)	60.9 (6.2)	68.5 (6.3)
APLL; 1	40.3 (5.3)	47.0 (5.6)	54.0 (6.0)	61.3 (6.3)	68.8 (6.6)
APLL; 2	44.1 (4.7)	49.2 (5.0)	54.4 (5.3)	59.5 (5.7)	64.7 (6.0)
APLL; 10	50.9 (3.9)	53.0 (4.1)	54.7 (4.2)	56.2 (4.4)	57.5 (4.6)
APLL; 20	52.6 (3.7)	53.8 (3.9)	54.7 (4.0)	55.5 (4.1)	56.2 (4.2)
APLL; 30	53.2 (3.7)	54.0 (3.8)	54.7 (3.9)	55.2 (3.9)	55.7 (4.0)
SLLF	53.5 (3.3)	54.3 (3.4)	54.8 (3.5)	55.4 (3.6)	55.9 (3.6)
t=200					
SFCFS	52.4 (5.3)	60.6 (5.7)	68.5 (5.8)	76.9 (6.1)	84.4 (6.1)
SPLL; 1	52.6 (5.3)	60.6 (5.6)	68.0 (5.7)	76.8 (6.0)	84.9 (6.4)
APLL; 1	52.6 (5.3)	60.5 (5.6)	68.6 (5.8)	76.9 (6.0)	85.3 (6.3)
APLL; 2	57.2 (4.6)	63.1 (4.9)	69.1 (5.2)	75.0 (5.5)	80.9 (5.8)
APLL; 10	65.3 (3.6)	67.6 (3.8)	69.6 (4.0)	71.3 (4.2)	72.9 (4.4)
APLL; 20	67.0 (3.5)	68.3 (3.6)	69.4 (3.8)	70.3 (3.9)	71.2 (4.0)
APLL; 30	68.0 (3.4)	68.9 (3.5)	69.7 (3.6)	70.3 (3.7)	70.8 (3.7)
SLLF	68.7 (3.1)	69.2 (3.1)	69.6 (3.1)	70.1 (3.2)	70.6 (3.2)
t=300					
SFCFS	56.3 (4.9)	63.9 (5.5)	72.3 (5.7)	80.4 (5.8)	88.8 (6.4)
SPLL; 1	55.9 (5.3)	63.8 (5.3)	72.2 (5.6)	80.5 (5.8)	89.[illegible] (6.3)
APLL; 1	56.2 (5.3)	64.2 (5.5)	72.5 (5.7)	80.9 (5.9)	89.5 (6.1)
APLL; 2	60.9 (4.6)	66.9 (4.8)	73.0 (5.1)	79.0 (5.4)	85.0 (5.7)
APLL; 10	69.3 (3.5)	71.6 (3.7)	73.6 (3.9)	75.5 (4.1)	77.1 (4.3)
APLL; 20	71.2 (3.3)	72.5 (3.5)	73.7 (3.6)	74.6 (3.7)	75.5 (3.8)
APLL; 30	72.1 (3.3)	73.0 (3.4)	73.8 (3.5)	74.5 (3.6)	75.0 (3.6)
SLLF	72.6 (3.0)	73.1 (3.0)	73.6 (3.1)	74.0 (3.1)	74.5 (3.2)

Figure 14a. Example 5.1 Results Summary: means (standard deviations)

Case	$N_1(t)$	$N_2(t)$	$N_3(t)$	$N_4(t)$	$N_5(t)$
t=400					
SFCFS	56.9 (5.0)	64.9 (5.4)	73.3 (5.9)	81.8 (5.4)	89.7 (6.1)
SPLL; 1	57.3 (5.1)	64.6 (5.4)	73.8 (5.6)	82.0 (5.8)	89.9 (5.8)
APLL; 1	57.2 (5.2)	65.3 (5.5)	73.6 (5.7)	82.0 (5.9)	90.5 (6.1)
APLL; 2	61.9 (4.5)	67.9 (4.8)	74.0 (5.1)	80.1 (5.4)	86.1 (5.6)
APLL; 10	70.5 (3.5)	72.8 (3.7)	74.8 (3.9)	76.6 (4.1)	78.3 (4.3)
APLL; 20	72.4 (3.3)	73.8 (3.5)	74.9 (3.6)	75.9 (3.7)	76.7 (3.8)
APLL; 30	73.2 (3.2)	74.2 (3.4)	74.9 (3.5)	75.6 (3.6)	76.2 (3.6)
SLLF	73.9 (3.0)	74.4 (3.0)	74.8 (3.0)	75.2 (3.0)	75.7 (3.1)
t=500					
SFCFS	57.2 (5.1)	65.5 (5.6)	73.7 (6.0)	81.6 (5.9)	90.3 (6.0)
SPLL; 1	56.8 (5.1)	65.5 (5.3)	73.5 (5.4)	82.0 (5.7)	90.3 (5.9)
APLL; 1	57.5 (5.2)	65.6 (5.5)	73.8 (5.7)	82.2 (5.9)	90.8 (6.0)
APLL; 2	62.2 (4.5)	68.2 (4.8)	74.3 (5.1)	80.4 (5.4)	86.5 (5.6)
APLL; 10	70.8 (3.4)	73.1 (3.7)	75.2 (3.9)	77.0 (4.1)	78.7 (4.3)
APLL; 20	72.8 (3.3)	74.1 (3.4)	75.2 (3.6)	76.2 (3.7)	77.1 (3.8)
APLL; 30	73.5 (3.2)	74.5 (3.4)	75.3 (3.5)	75.9 (3.5)	76.5 (3.6)
SLLF	74.0 (3.2)	74.5 (3.3)	74.9 (3.3)	75.3 (3.3)	75.8 (3.3)
t=600					
SFCFS	57.4 (5.0)	65.6 (5.3)	73.5 (5.5)	82.3 (5.8)	90.4 (6.0)
SPLL; 1	57.5 (5.1)	65.7 (5.5)	73.7 (5.6)	82.2 (5.8)	91.0 (6.0)
APLL; 1	57.6 (5.2)	65.6 (5.4)	73.9 (5.7)	82.3 (5.9)	90.9 (6.0)
APLL; 2	62.3 (4.5)	68.3 (4.8)	74.4 (5.1)	80.5 (5.4)	86.6 (5.6)
APLL; 10	70.9 (3.4)	73.2 (3.7)	75.3 (3.9)	77.1 (4.1)	78.8 (4.3)
APLL; 20	72.9 (3.3)	74.2 (3.4)	75.3 (3.6)	76.3 (3.7)	77.2 (3.8)
APLL; 30	73.6 (3.2)	74.6 (3.3)	75.4 (3.5)	76.0 (3.6)	76.6 (3.6)
SLLF	74.2 (2.9)	74.6 (3.0)	75.1 (3.1)	75.5 (3.1)	75.9 (3.1)
t=700					
SFCFS	57.4 (4.8)	65.7 (5.5)	73.6 (5.9)	82.2 (6.0)	90.8 (5.8)
SPLL; 1	57.2 (5.3)	65.3 (5.6)	73.7 (5.9)	81.8 (5.5)	90.6 (6.3)
APLL; 1	57.6 (5.2)	65.7 (5.4)	73.9 (5.7)	82.4 (5.9)	90.9 (6.0)
APLL; 2	62.3 (4.5)	68.4 (4.8)	74.4 (5.1)	80.5 (5.4)	86.6 (5.6)
APLL; 10	70.9 (3.4)	73.2 (3.7)	75.3 (3.9)	77.1 (4.1)	78.8 (4.3)
APLL; 20	72.9 (3.3)	74.2 (3.4)	75.4 (3.6)	76.3 (3.7)	77.2 (3.9)
APLL; 30	73.7 (3.2)	74.6 (3.3)	75.4 (3.5)	76.0 (3.5)	76.6 (3.6)
SLLF	74.3 (3.2)	74.8 (3.2)	75.2 (3.2)	75.7 (3.3)	76.1 (3.3)

Figure 14b. Example 5.1 Results Summary: means (standard deviations) (cont.)

Discussion of the Tabulated Results: At all t, the results show good agreement between cases SPLL;1 and APLL;1, i.e., the diffusion approximation yields solutions close to the results from the simulation with probabilistic service. There is also good agree-

ment between cases APLL;30 and SLLF; i.e., the diffusion approximation with a high power of p yields solutions close to the results from the simulation with service of the longest line first. The longest-line-first discipline tends to drive the items toward equal queue lengths. This makes intuitive sense since whenever the number of items awaiting repair for one particular item exceeds the number awaiting repair for the other items, it gets preferential service. At each time shown in the results, the effect of increasing the power p is seen to move the diffusion approximation results toward the LLF results.

Sample data were taken from the simulation in Case SPLL;1 of Example 5.1 to assess the validity of the assumption of normality underlying the heavy traffic model at times when the system was in transient and steady-state phases.

For a transient phase time, $t = 50$, an empirical histogram of the data for one of the item types is shown in Figure 15, with a Normal density overlaid on the histogram. Also shown is a Normal probability (quantile-quantile) plot. The chi-square goodness of fit test for this example yielded a test statistic of 5.336, with 5 degrees of freedom, and a significance level of 0.376, i.e., no significant departure from normality.

For a steady-state time, $t = 700$, the histogram, normal density and probability plots are shown in Figure 16. At $t = 700$, the chi-square goodness of fit test for this example yielded a test statistic of 37.2, with 6 degrees of freedom, and a significance level of 1.6×10^{-6}, i.e., statistically significant departure from normality. However, the probability plot shows good agreement from the first through the 99th percentiles, and consequently, the diffusion approximation does yield good agreement with the simulation results.

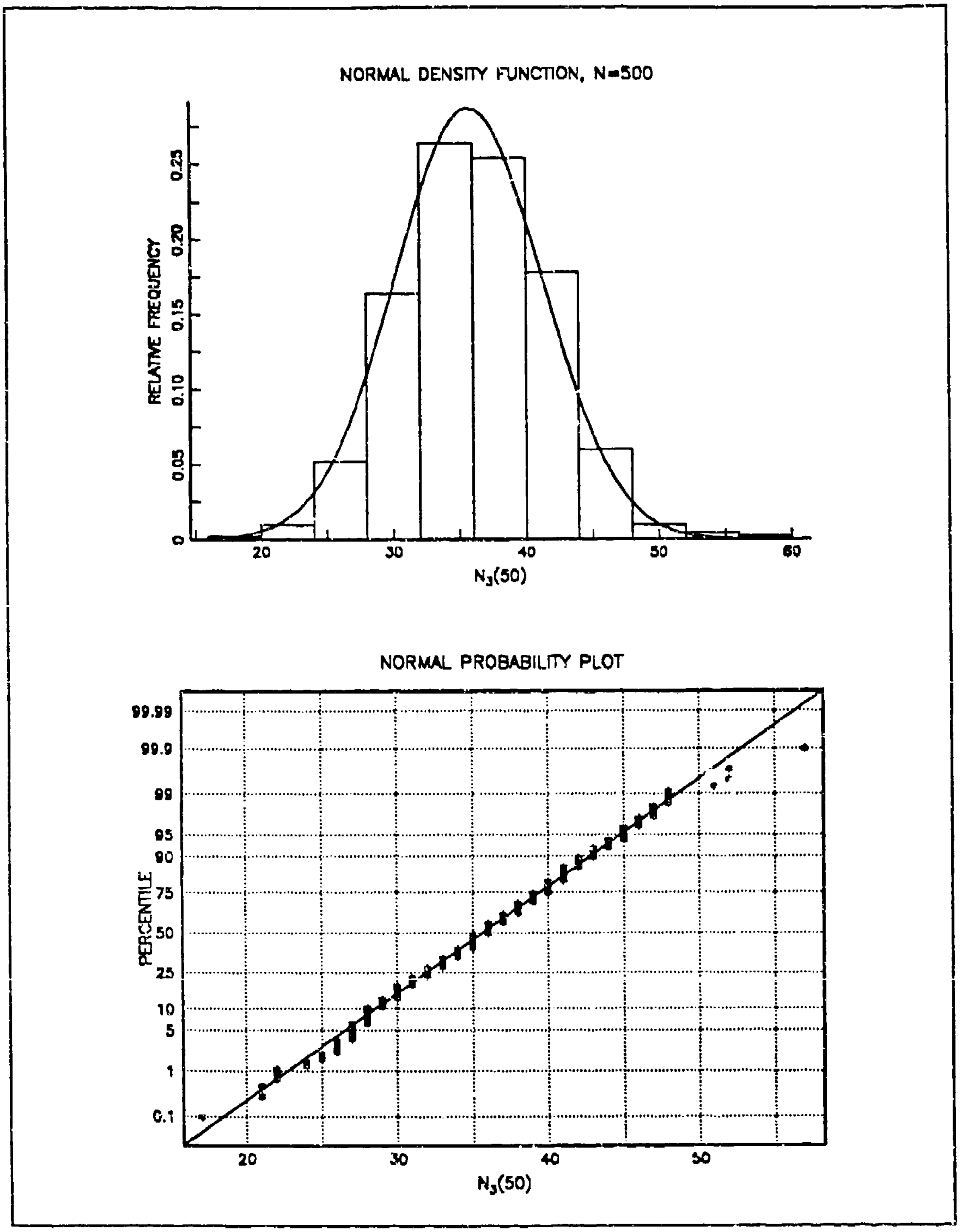


Figure 15. Example 5.1 Queue Length Normality (transient)

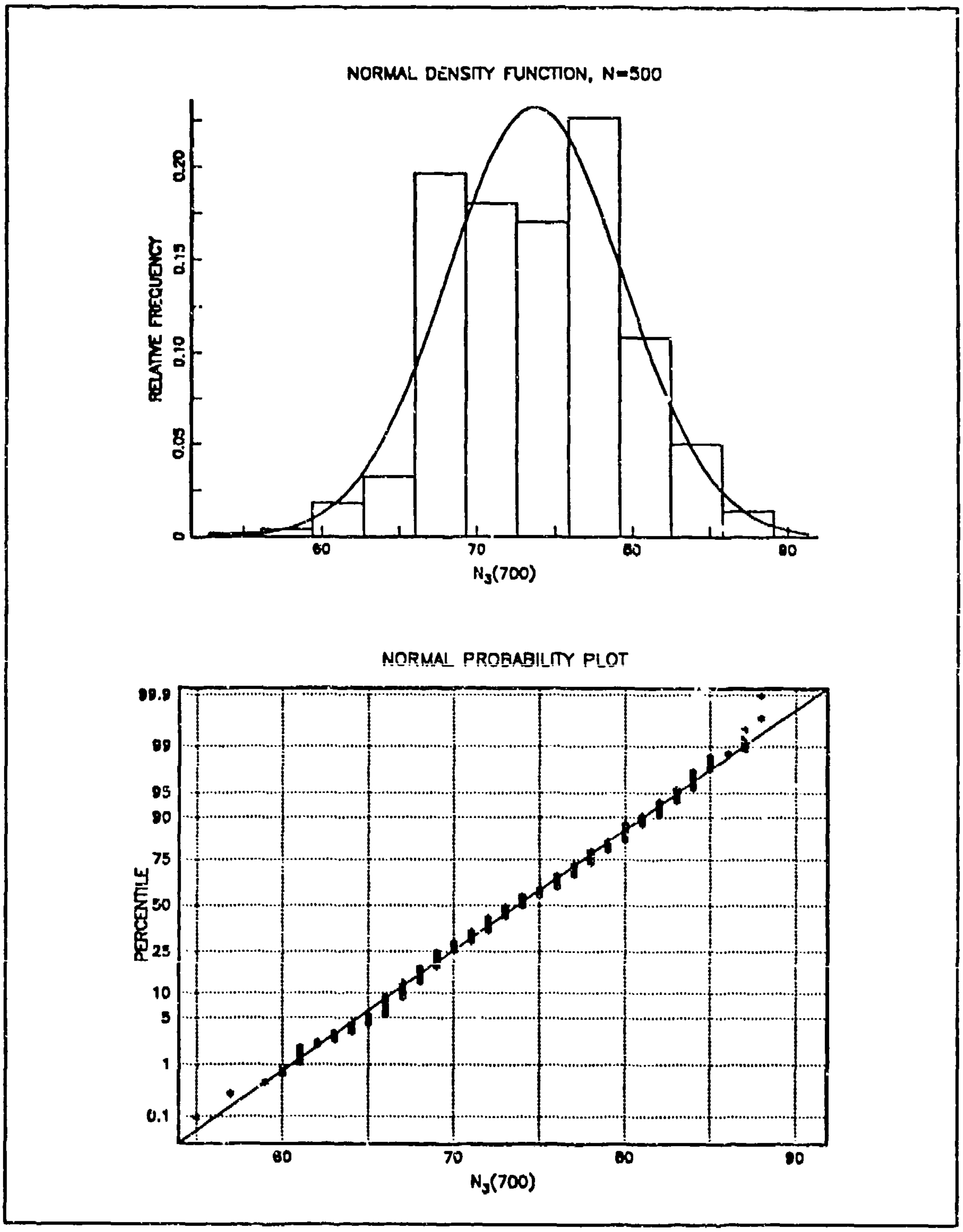


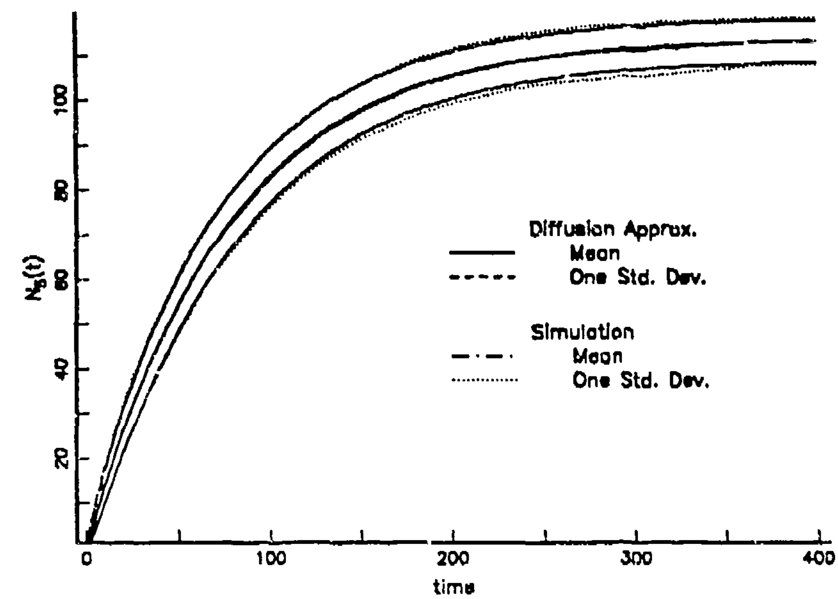
Figure 16. Example 5.1 Queue Length Normality (steady-state)

Example 5.2: In this example, several cases are presented to demonstrate the results using the dynamic-service-selection model, using different inputs (especially different service rates). The inputs for this example are shown in Figure 17. All cases in this example use the same repair policy -- PLL;1 -- service priority proportional to queue-length ($w_i = 1$, for all i, and $p = 1$).

	i	1	2	3	4	5
Case 1						
	K_i	100	110	120	130	140
	λ_i	.013	.013	.013	.013	.013
Case 2						
	K_i	50	100	150	200	250
	λ_i	.013	.013	.013	.013	.013
Case 3						
	K_i	100	110	120	130	140
	λ_i	.01	.02	.03	.02	.01
Case 4						
	K_i	50	100	150	200	250
	λ_i	.01	.02	.03	.02	.01
All Cases						
	v_i	.500	1.00	3.00	4.00	4.50
	w_i	1.00	1.00	1.00	1.00	1.00
	$N(0)$	0	0	0	0	0

Figure 17. Example 5.2 Inputs

Transient Results: For each case in Example 5.2, a typical resulting queue length as a function of time is shown in Figure 18. Results are shown for the queue developed for one of the items, comparing the diffusion approximation differential equation solution with the corresponding simulation results. Queue length means and standard deviations were computed at unit time steps.



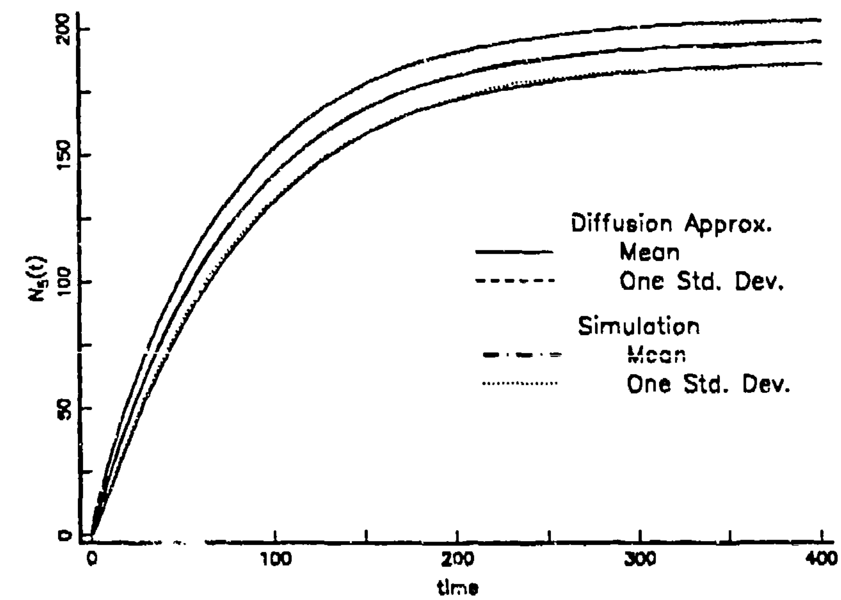


Figure 18a. Example 5.2, Cases 1 and 2, Queue Length vs. Time

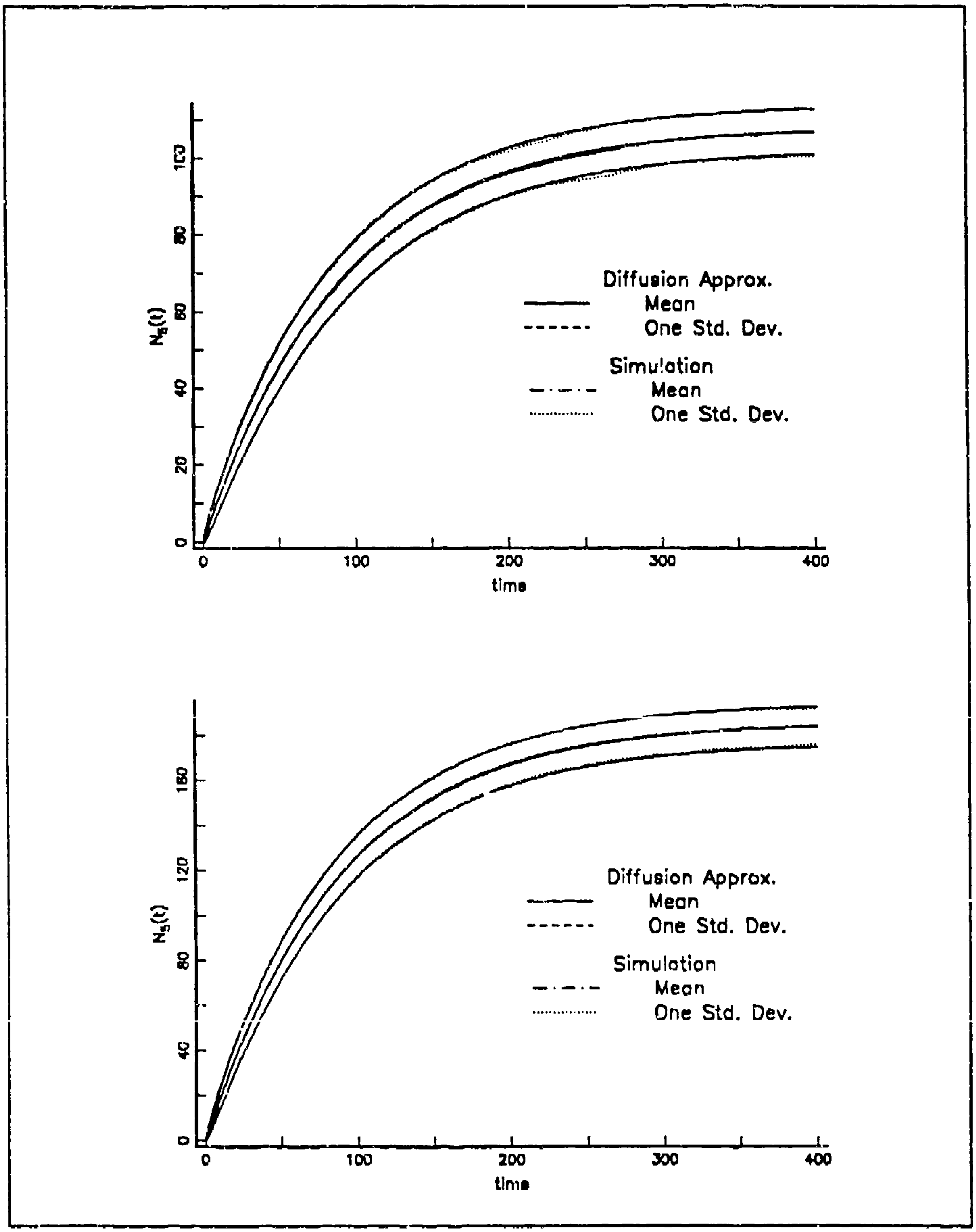


Figure 18b. Example 5.2, Cases 3 and 4, Queue Length vs. Time

Steady-state Results: For each case in Example 5.2, steady-state numerical results are summarized in Figure 19. Also shown are 95 percent confidence intervals for the estimates obtained from the simulation, which were based upon steady-state observations from time 1001 through time 2000, using the method of batch means, with ten batches each of length 100; see Welch [Ref. 46].

i	1	2	3	4	5
Means:					
Diff. Approx. (ODE)	81.40	89.54	97.68	105.82	113.96
Sim.	81.14	89.39	97.47	105.60	113.74
Sim. .95CI lower	81.04	89.23	97.37	105.42	113.53
Sim. .95CI upper	81.24	89.54	97.57	105.79	113.95
Standard Deviations:					
Diff. Approx. (SDE)	3.75	4.38	4.91	5.20	5.47
Morrison(gen. fcn.)	3.73	4.38	4.93	5.23	5.50
Sim.	3.75	4.38	4.89	5.24	5.56
Sim. .95CI lower	3.67	4.31	4.80	5.17	5.48
Sim. .95CI upper	3.82	4.45	4.97	5.31	5.65

Figure 19a. Example 5.2 Case 1 Steady-state Summary

i	1	2	3	4	5
Means:					
Diff. Approx. (ODE)	39.18	78.37	117.55	156.73	195.91
Sim.	39.10	78.17	117.35	156.26	195.57
Sim. .95CI lower	39.03	78.05	117.12	155.95	195.30
Sim. .95CI upper	39.18	78.28	117.57	156.57	195.84
Standard Deviations:					
Diff. Approx. (SDE)	2.85	4.33	5.90	7.24	8.56
Morrison(gen. fcn.)	2.78	4.25	5.85	7.13	8.34
Sim.	2.78	4.21	5.90	7.19	8.33
Sim. .95CI lower	2.75	4.16	5.81	7.13	8.26
Sim. .95CI upper	2.80	4.27	6.00	7.25	8.40

Figure 19b. Example 5.2 Case 2 Steady-state Summary

i	1	2	3	4	5
Means:					
Diff. Approx. (ODE)	77.21	95.86	109.26	113.29	108.09
Sim.	76.93	95.65	108.99	113.05	107.79
Sim. .95CI lower	76.80	95.53	108.91	112.90	107.66
Sim. .95CI upper	77.07	95.76	109.07	113.21	108.92
Standard Deviations:					
Diff. Approx. (SDE)	4.07	3.73	3.59	4.47	5.84
Sim.	4.06	3.77	3.61	4.49	5.88
Sim. .95CI lower	4.01	3.72	3.56	4.43	5.79
Sim. .95CI upper	4.12	3.82	3.66	4.54	5.97

Figure 19c. Example 5.2 Case 3 Steady-state Summary

i	1	2	3	4	5
Means:					
Diff. Approx. (ODE)	37.11	85.21	134.44	170.42	185.56
Sim.	36.99	85.00	134.25	170.21	185.26
Sim. .95CI lower	36.90	84.87	134.11	170.05	185.02
Sim. .95CI upper	37.07	85.13	134.40	170.37	185.49
Standard Deviations:					
Diff. Approx. (SDE)	3.03	3.65	4.31	6.14	8.93
Sim.	3.02	3.59	4.27	6.07	8.63
Sim. .95CI lower	2.97	3.52	4.19	6.03	8.46
Sim. .95CI upper	3.06	3.66	4.34	6.11	8.80

Figure 19d. Example 5.2 Case 4 Steady-state Summary

Normality Analysis: Sample data taken from the simulation in Case 1 of Example 5.2 to assess the validity of the assumption of normality underlying the heavy traffic model during the transient phase (time 50) and in steady-state (time 700) are plotted in histograms in Figure 20and Figure 21, respectively. Normal densities are overlaid on the histograms, and Normal probability (quantile-quantile) plots are shown.

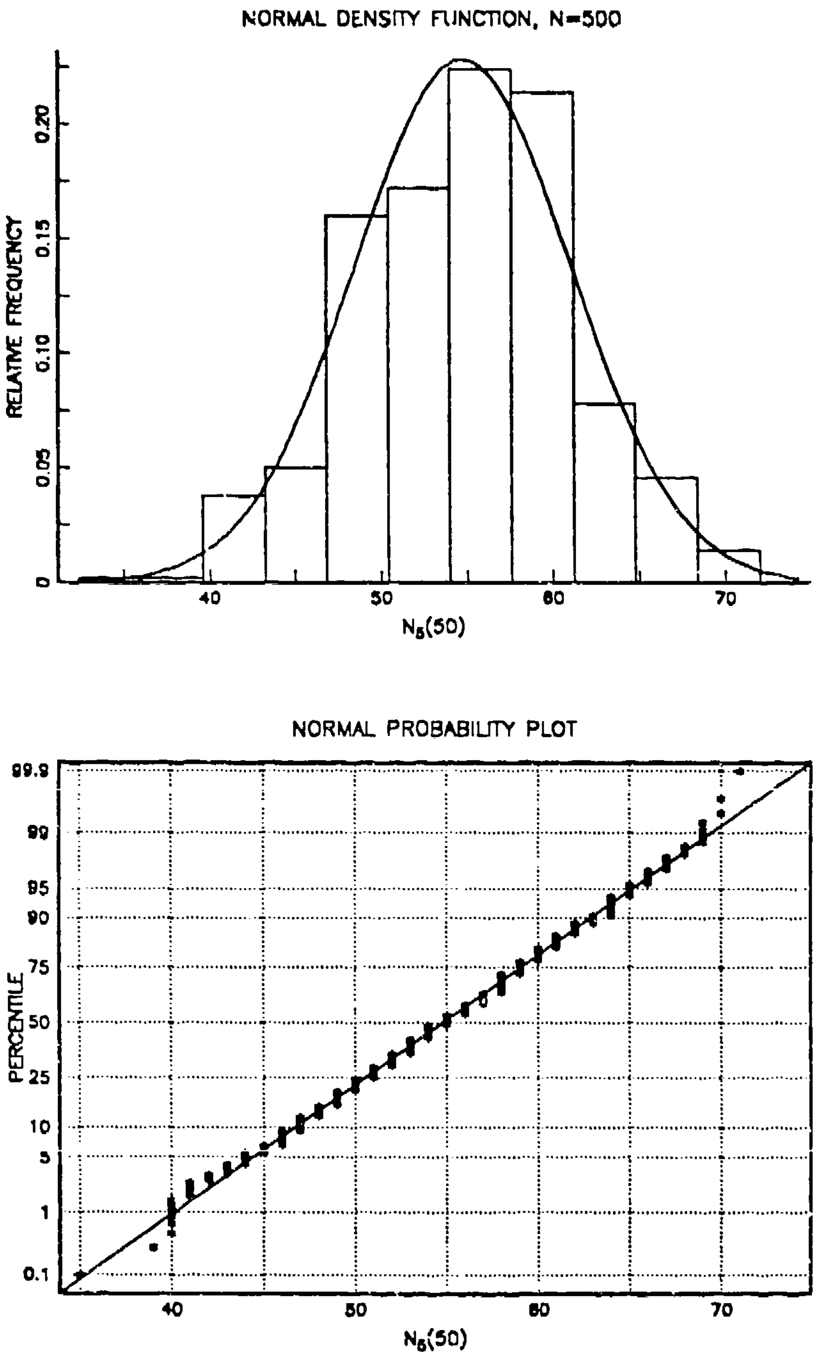


Figure 20. Example 5.2 Case 1 Queue Length Normality (transient)

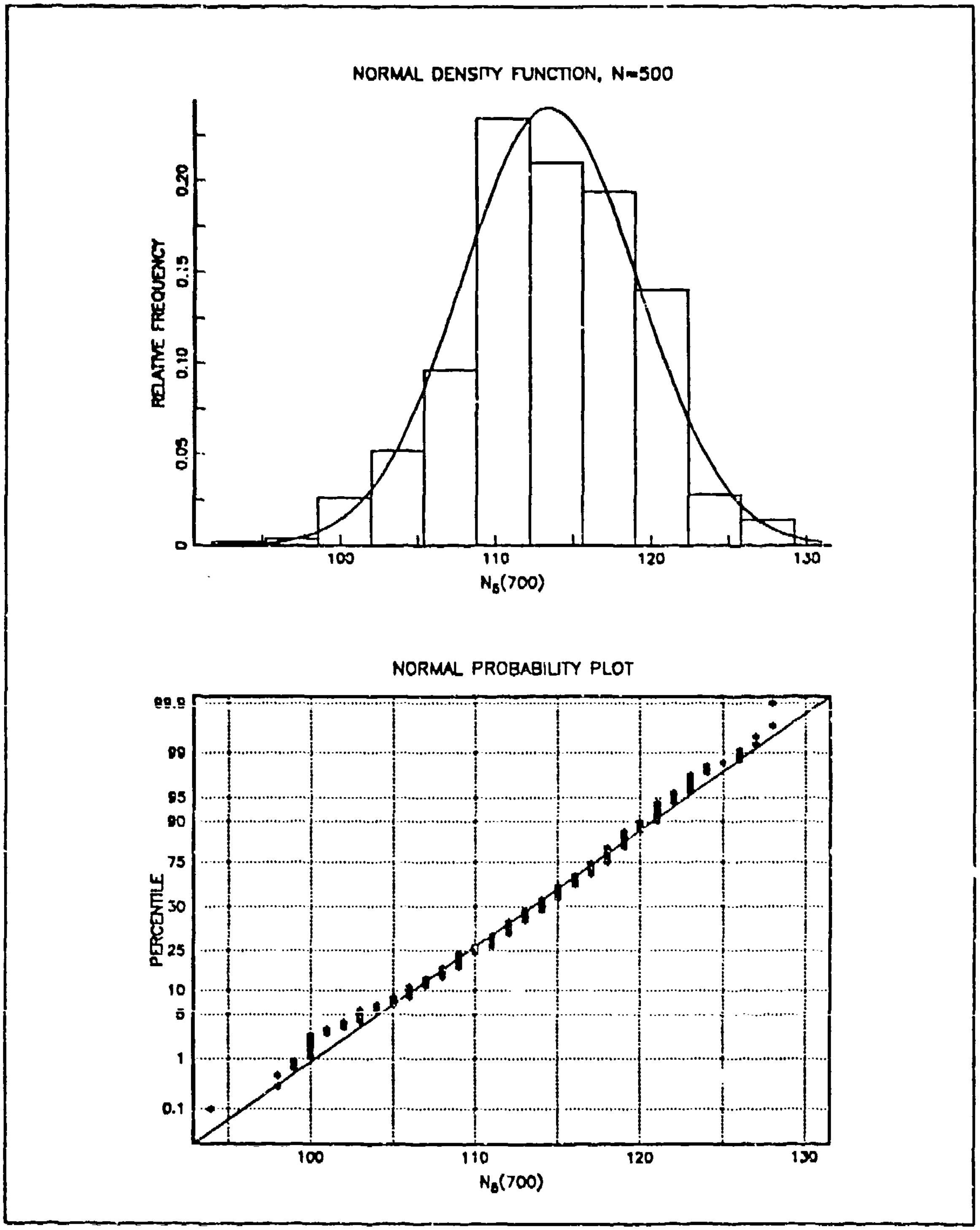


Figure 21. Example 5.2 Case 1 Queue Length Normality (steady-state)

Example 5.3: In this example, several cases are presented to demonstrate the results using the probabilistic-lowest-availability service discipline with the dynamic-service-selection model. In addition to the lowest-availability cases, cases with probabilistic-longest-line service and first-come-first-served service disciplines are presented for contrast. The inputs for this example are shown in Figure 22.

i	1	2	3	4	5
K_i	100	110	120	130	140
λ_i	0.015	0.020	0.025	0.030	0.035
v_i	1.0	1.1	1.2	1.3	1.4
w_i	1.0	1.0	1.0	1.0	1.0
$N(0)$	0	0	0	0	0

Figure 22. Example 5.3 Inputs

Results for Example 5.3 are obtained and presented for each of the following cases.

1. Probabilistic-lowest-availability (PLA;1) Cases.

1a. Case APLA;1 (Diffusion Approximation, **PLA;1** Service). This case is the numerical solution obtained from the diffusion approximation in which the service rule is modeled by the probabilistic form

$$q_i(\mathbf{N}(t)) = \frac{w_i\,(K_i - N_i(t))^{-p}}{\sum_j w_j\,(K_j - N_j(t))^{-p}} \quad ; \tag{5.43}$$

with the parameter p set equal to 1.

1b. Case SPLA;1 (Simulation, **PLA;1** Service). This case is the simulation outcome in which the service discipline is randomized selection of the next queue for service, upon each service completion, in accordance with probabilities using (5.43).

2. Lowest-availability-first (LAF) Cases.

2a. Case APLA;10 (Diffusion Approximation, **PLA;10** Service). This case is the is the same as Case APLA;1, but with the parameter p set equal to a high value, in

this example 10, to get an analytical solution which approximates service of the item with the lowest availability first (LAF).

2b. Case SLAF (Simulation, **LAF** Service). This case is the simulation outcome in which the service discipline is to select the item with the lowest availability to receive the next service upon each service completion.

3. First-come-first-served (FCFS) Cases.

3a. Case APLL;1 (Diffusion Approximation, **PLL;1** Service). This case is the numerical solution obtained from the diffusion approximation in which the service discipline is probabilistic-longest-line with the parameter p set equal to 1, which approximates FCFS.

3b. Case SFCFS (Simulation, **FCFS** Service). This case is the simulation outcome in which the service discipline is first-come-first-serve.

Transient Results: For Example 5.3, the transient responses of the system for all cases are summarized in tabular form in Figure 23a. and b.. Since the service discipline is based on the availability of each item, the output in this example shows the number operational instead of the numbers in the queue for repair as in the previous examples. Means and standard deviations of the availability of each item as a function of time are given at selected times. Standard errors and confidence intervals for the point estimates for the means and standard deviations obtained from the simulation are omitted from the tabulated results to avoid more clutter in the table. Upper and lower .95 confidence limits for the means are the point estimate $\pm$.0877 times the corresponding estimate for the standard deviation (i.e., about $\pm$ 10% of the standard deviation). Upper and lower .95 confidence limits for the standard deviations are .942 and 1.066 times the point estimate (i.e., about $\pm$ 5%). Following the tabulated results the transient availability of one of the items is displayed graphically as a function of time in Figure 24a., b. and c.. The plots show the solutions for both the means and standard deviations for each corresponding pair of cases to compare diffusion approximation results with corresponding simulation results. Mean item availability and standard deviations were computed at unit time steps.

Item:	Means: 1	2	3	4	5	Standard Deviations: 1	2	3	4	5
t = 50										
APLA; 1	54.8	47.5	41.0	35.4	30.7	5.4	5.4	5.3	5.0	4.7
SPLA; 1	55.0	47.5	41.1	36.1	31.0	5.5	5.4	5.6	4.7	4.9
APLA; 10	52.2	44.9	40.1	37.5	36.0	4.8	4.4	3.9	3.6	3.4
SLAF	51.3	43.9	40.1	38.4	37.5	4.7	4.3	3.4	3.2	3.4
APLL; 1	52.1	46.5	41.4	36.8	32.7	5.1	5.3	5.3	5.2	5.1
SFCFS	51.5	46.1	41.0	37.0	33.3	5.0	5.1	5.2	5.5	5.2
t = 100										
APLA; 1	31.3	23.6	18.6	15.5	13.4	4.6	4.1	3.6	3.2	2.7
SPLA; 1	31.5	23.8	18.8	15.6	13.7	4.6	4.3	3.5	3.2	3.0
APLA; 10	25.4	20.4	19.3	18.8	18.4	3.9	2.7	2.5	2.5	2.5
SLAF	24.5	20.0	19.2	18.6	18.1	3.9	2.2	2.0	2.1	2.1
APLL; 1	30.3	23.7	19.0	15.7	13.4	4.6	4.3	4.0	3.6	3.2
SFCFS	28.9	22.8	18.6	16.3	14.5	4.6	4.5	4.2	3.7	3.9
t = 150										
APLA; 1	20.1	15.1	12.5	11.0	10.0	3.7	3.1	2.8	2.6	2.7
SPLA; 1	20.2	15.3	12.8	11.2	10.1	3.8	3.0	2.8	2.6	2.5
APLA; 10	14.5	13.7	13.3	13.0	12.8	2.2	2.1	2.0	2.0	2.0
SLAF	14.3	13.4	13.0	12.7	12.3	1.9	1.7	1.7	1.8	1.8
APLL; 1	20.5	15.4	12.4	10.5	9.4	4.1	3.7	3.4	3.2	3.1
SFCFS	18.3	14.3	12.3	11.3	11.2	3.9	3.5	3.4	3.3	3.5
t = 200										
APLA; 1	15.4	12.3	10.7	9.7	8.9	3.1	2.7	2.6	2.4	2.4
SPLA; 1	15.5	12.5	10.8	9.9	9.1	3.2	2.8	2.6	2.5	2.4
APLA; 10	11.5	11.2	10.9	10.7	10.6	1.9	1.9	1.9	1.8	1.8
SLAF	11.7	11.3	11.0	10.7	10.4	1.6	1.6	1.7	1.6	1.7
APLL; 1	16.1	12.5	10.5	9.3	8.5	3.7	3.3	3.1	3.0	2.9
SFCFS	13.9	11.6	10.7	10.2	10.4	3.3	3.4	3.3	3.2	3.3

Figure 23a. Example 5.3: Number Operational (Transient)

	Means:					Standard Deviations:				
Item:	1	2	3	4	5	1	2	3	4	5
t = 250										
APLA; 1	13.7	11.4	10.1	9.2	8.5	2.9	2.6	2.5	2.4	2.3
SPLA; 1	13.7	11.7	10.4	9.4	8.6	2.8	2.7	2.6	2.4	2.3
APLA; 10	10.7	10.4	10.2	10.0	9.9	1.9	1.9	1.8	1.8	1.7
SLAF	11.0	10.6	10.3	10.0	9.6	1.6	1.6	1.6	1.7	1.6
APLL; 1	14.2	11.4	9.9	9.0	8.3	3.5	3.2	3.0	2.9	2.8
SFCFS	11.8	10.9	10.0	10.2	10.0	3.1	3.0	3.0	3.0	2.9
t = 300										
APLA; 1	13.0	11.1	9.9	9.0	8.3	2.8	2.6	2.4	2.3	2.2
SPLA; 1	13.1	11.1	10.2	9.2	8.4	2.8	2.7	2.5	2.2	2.3
APLA; 10	10.4	10.1	9.9	9.8	9.6	1.9	1.8	1.8	1.8	1.8
SLAF	10.8	10.5	10.1	9.7	9.5	1.5	1.5	1.6	1.5	1.6
APLL; 1	13.4	11.1	9.8	8.9	8.3	3.4	3.2	3.0	2.9	2.9
SFCFS	11.5	10.7	9.9	9.9	9.1	3.1	3.3	3.0	3.1	2.9
t = 350										
APLA; 1	12.7	11.0	9.8	8.9	8.3	2.7	2.6	2.4	2.3	2.3
SPLA; 1	13.0	11.1	10.1	9.2	8.4	2.9	2.6	2.4	2.4	2.4
APLA; 10	10.3	10.0	9.8	9.7	9.5	1.9	1.8	1.8	1.8	1.8
SLAF	10.9	10.5	10.2	9.9	9.7	1.5	1.6	1.6	1.6	1.6
APLL; 1	13.1	11.0	9.7	8.9	8.3	3.4	3.2	3.0	2.9	2.8
SFCFS	11.9	10.9	10.2	9.4	8.7	3.3	3.1	3.1	3.0	3.0
t = 400										
APLA; 1	12.6	10.9	9.8	8.9	8.2	2.7	2.5	2.4	2.3	2.2
SPLA; 1	13.0	11.2	10.1	9.1	8.5	2.7	2.6	2.5	2.4	2.2
APLA; 10	10.3	10.0	9.8	9.7	9.5	1.9	1.8	1.8	1.8	1.9
SLAF	10.6	10.4	10.0	9.7	9.5	1.6	1.5	1.5	1.6	1.6
APLL; 1	12.9	10.9	9.7	8.9	8.3	3.3	3.1	3.0	2.9	2.9
SFCFS	12.2	11.2	10.1	9.2	8.0	3.4	3.0	3.1	3.0	2.9

Figure 23b. Example 5.3: Number Operational (Transient)

Discussion of the Tabulated Results: At all t, the results show that the diffusion approximation yields solutions close to the results from the corresponding simulation in

both mean and standard deviation. Looking across the rows for all cases at all times, item 5 has the lowest availability. It may be seen that since it gets the most preferential service under LAF, the availability of item 5 drops the least rapidly under LAF than under PLA;1 or FCFS. However, the preferential treatment of item 5 is at the expense of item 1 which has the highest availability at all times. Consequently, item 1 availability is dropping the most rapidly under LAF than under PLA;1 or FCFS. Looking at the spread in means across items, it is seen that LAF tends to drive the item availabilities toward some average value. The standard deviations across the items, at all times, in all cases are fairly consistent. At all times the standard deviations under LAF are the lowest of the three cases, and FCFS the highest. The lowest standard deviations occurring under LAF is anticipated since that service discipline selects the next item for service *deterministically*. At times greater than about 150, the diffusion approximation standard deviation under APLA;10 is systematically higher than the simulation under SLAF. It is conjectured that the explanation for this is that even with as high a power as 10, APLA;10 is still a probabilistic service selection which inherently has more variation than LAF.

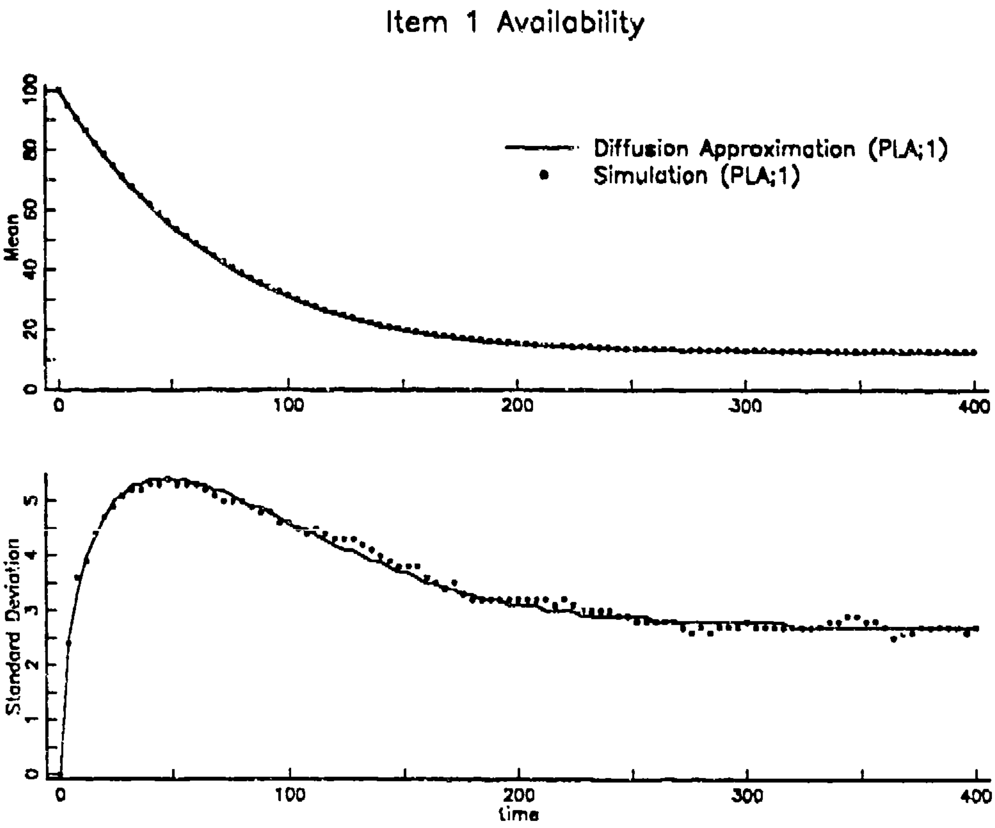


Figure 24a. Example 5.3 PLA;1: Item 1 Number Operational

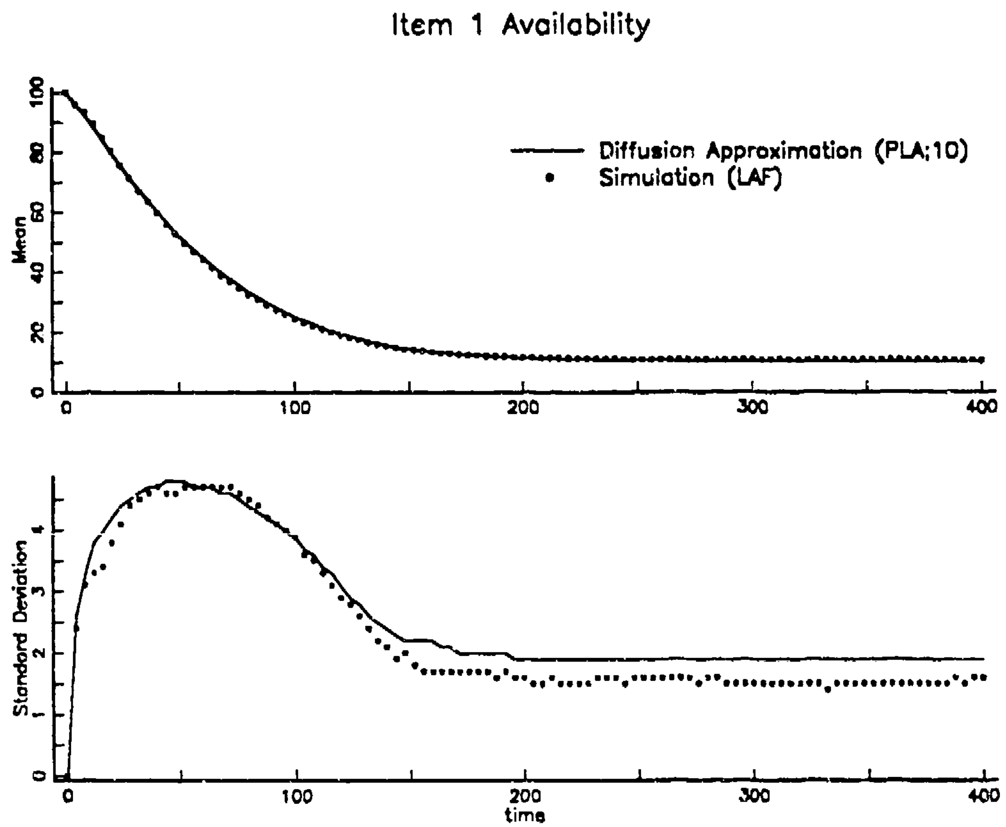


Figure 24b. Example 5.3 LAF: Item 1 Number Operational

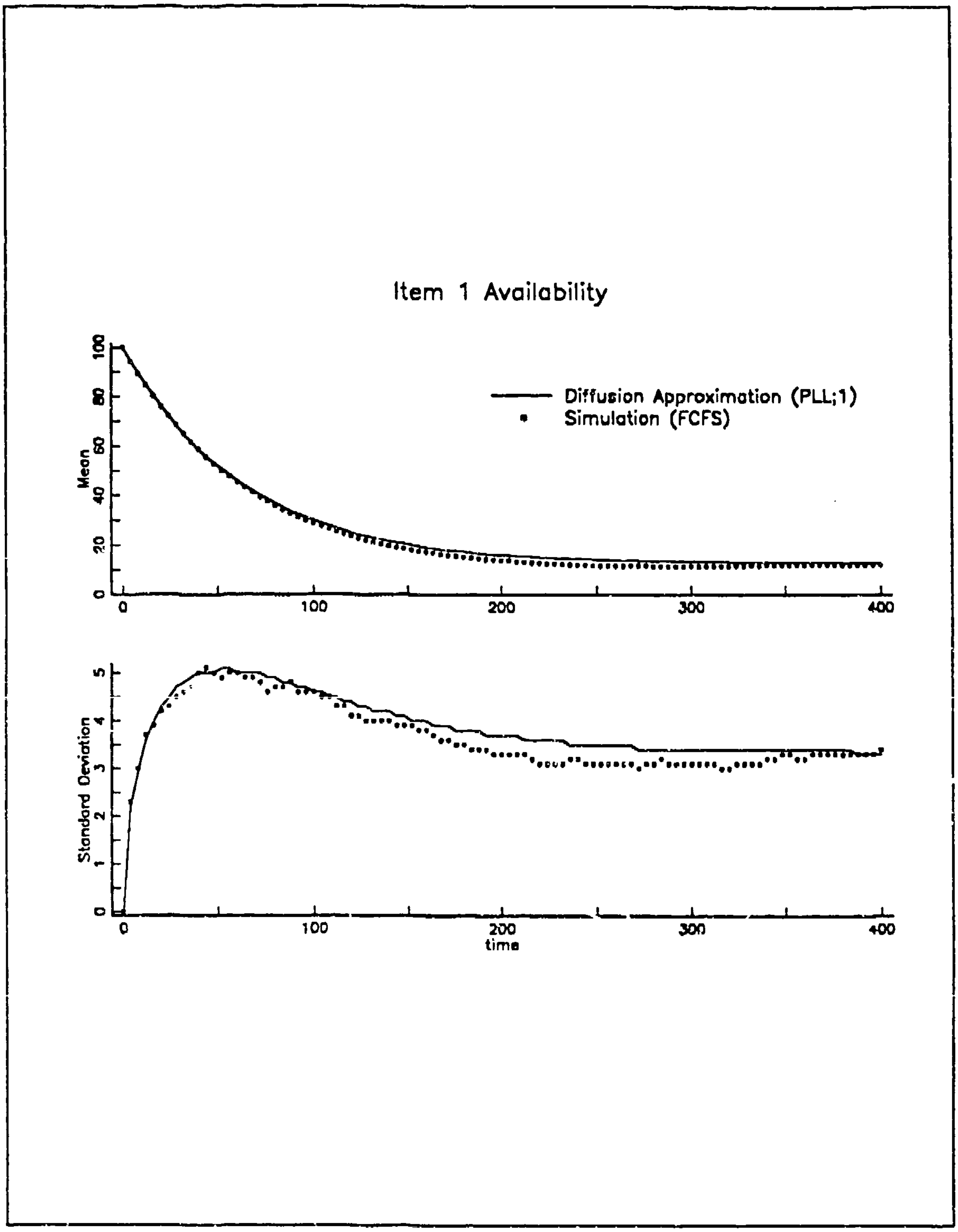


Figure 24c. Example 5.3 FCFS: Item 1 Number Operational

Steady-state Results: For Example 5.3, steady-state numerical results are summarized in Figure 25a., b. and c.. As observed in the transient results, it is seen that LAF tends to drive the extreme item availabilities toward some average value. Comparing the steady-state results in PLA;1 and FCFS, it is seen that the means are close, but the standard deviations in FCFS are consistently higher. As in the transient results, the standard deviations under LAF are lower than either PLA;1 or FCFS.

Means:

Item	APLA; 1	SPLA; 1	.95 C.I.
1	12.57	12.80	12.71, 12.89
2	10.88	11.14	11.07, 11.22
3	9.74	9.99	9.90, 10.08
4	8.89	9.14	9.14, 9.14
5	8.23	8.48	8.48, 8.48

Standard Deviations:

Item	APLA; 1	SPLA; 1	.95 C.I.
1	2.71	2.69	2.66, 2.71
2	2.54	2.56	2.53, 2.59
3	2.42	2.45	2.43, 2.48
4	2.32	2.36	2.34, 2.38
5	2.22	2.29	2.27, 2.31

Figure 25a. Example 5.3 PLA;1: Number Operational (Steady-state)

Means:

Item	APLA; 10	SLAF	.95 C.I.
1	10.27	10.74	10.65, 10.83
2	10.00	10.38	10.33, 10.44
3	9.80	10.07	10.00, 10.15
4	9.64	9.78	9.73, 9.83
5	9.51	9.50	9.39, 9.60

Standard Deviations:

Item	APLA; 10	SLAF	.95 C.I.
1	1.88	1.52	1.50, 1.53
2	1.84	1.55	1.53, 1.57
3	1.81	1.56	1.55, 1.57
4	1.82	1.58	1.57, 1.60
5	1.89	1.60	1.59, 1.61

Figure 25b. Example 5.3 LAF: Number Operational (Steady-state)

Means:

Item	APLL; 1	SFCFS	.95 C.I.
1	12.74	13.00	12.61, 13.39
2	10.86	11.10	10.85, 11.35
3	9.67	9.93	9.79, 10.07
4	8.84	9.10	8.87, 9.33
5	8.24	8.52	8.24, 8.79

Standard Deviations:

Item	APLL; 1	SFCFS	.95 C.I.
1	3.33	3.39	3.31, 3.46
2	3.14	3.17	3.13, 3.22
3	3.01	3.03	3.00, 3.06
4	2.92	2.98	2.95, 3.00
5	2.80	2.90	2.84, 2.97

Figure 25c. Example 5.3 FCFS: Number Operational (Steady-state)

E. APPLICATION: BAYESIAN BOOTSTRAPPING

Since the solution of the system of differential equations is computationally fast, the diffusion approximation may be applied to a setting in which the failure rates and service times are not known exactly, but must be inferred from data.

The idea, which will be called *Parametric Bayesian Bootstrapping*, is summarized as follows; see Efron [Ref. 47], and Dalal, Fowlkes, and Hoadley [Ref. 48]. A non-informative prior distribution is assumed for each failure rate and service rate.[4] Suppose some data are gathered on actual times to failure and service times. Using the likelihood functions for the data, and the priors, posterior distributions for the failure rates and service rates are determined. This much is the Bayesian part of the procedure. Then the bootstrap is used. For each replication of the bootstrap, the posterior distributions are sampled (i.e., pseudo-random failure rates and service rates are generated from the posterior distributions) to obtain a set of inputs for the diffusion approximation. From the diffusion approximation, an estimate is computed for, say, the probability that the number of each item awaiting or undergoing repair at a particular time of interest exceeds some specified value. This estimate is actually a *conditional* value given the ran-

[4] Any prior distribution may be assumed. A reason for perhaps using a non-informative prior is that it favors no possible values for each rate over any other, thus relying the most on the data; see Berger [Ref. 49].

domly selected failure rate and service rate inputs. Sampling from the posterior for rates, and subsequent computation from the diffusion approximation are replicated to produce many such conditional estimates, which are then averaged to remove the condition on the uncertain failure and service rates. Note that it is the speed and ease of computation that is possible with the diffusion approximation that makes the above process feasible, particularly on small computers.

Bayes Posterior Distributions. The Bayes posterior distributions are developed in the standard manner; see Berger [Ref. 49]. Given data are

$$b_{i1}, b_{i2}, \dots, b_{iB_i} \quad \text{times between failures for item } i;\ i = 1, \dots, I \ ,$$

and

$$r_{i1}, r_{i2}, \dots, r_{iR_i} \quad \text{times to repair for item } i;\ i = 1, \dots, I \ .$$

Let the vectors of data be collectively denoted $\mathbf{b}_i$ and $\mathbf{r}_i$. Using the assumption that each item has Markovian failures at rate λ_i, and independent exponentially distributed service times with mean v_i^{-1}, the likelihood functions are

$$\begin{aligned} L(\lambda_i; \mathbf{b}_i) &= \prod_{j=1}^{B_i} \lambda_i e^{-\lambda_i b_{ij}} \\ &= \lambda_i^{B_i} e^{-\lambda_i b_{i+}} \ , \end{aligned}$$

and

$$\begin{aligned} L(v_i; \mathbf{r}_i) &= \prod_{j=1}^{R_i} v_i e^{-v_i r_{ij}} \\ &= v_i^{R_i} e^{-v_i r_{i+}} \ ; \end{aligned}$$

for $i = 1, \dots, I$. The product of each likelihood function and the respective non-informative prior, $\xi_i(\lambda_i) = \lambda_i^{-1}$, or $\psi_i(v_i) = v_i^{-1}$, is proportional to the Bayes posterior densities as follows:

$$\xi_i(\lambda_i; \mathbf{b}_i) \propto \lambda_i^{B_i} e^{-\lambda_i b_{i+}} \frac{1}{\lambda_i} \ ,$$

and

$$\psi_i(v_i; \mathbf{r}_i) \propto v_i^{R_i} e^{-v_i r_{i+}} \frac{1}{v_i} \quad .$$

Recognizing the form of the gamma density, the constants of proportionality are chosen to get the posterior density functions

$$\xi_i(\lambda_i; \mathbf{b}_i) = \frac{b_{i+}^{B_i}}{\Gamma(B_i)} \lambda_i^{B_i - 1} e^{-\lambda_i b_{i+}} \quad , \tag{5.44}$$

and

$$\psi_i(v_i; \mathbf{r}_i) = \frac{r_{i+}^{R_i}}{\Gamma(R_i)} v_i^{R_i - 1} e^{-v_i r_{i+}} \quad . \tag{5.45}$$

A Measure of Effectiveness. In general, the bootstrap can be used to obtain an estimate of any computable function of the failure and service rates, which will be denoted $\theta(\underline{\lambda}, \underline{v})$. To illustrate how the results obtained from the diffusion approximation might be used, the following measure of effectiveness will be considered. Suppose it is of interest if the operable number of item i at time τ is below some critical value x_i. An appropriate measure of effectiveness then is

$$\theta(\underline{\lambda}, \underline{v}) = \mathrm{P}\{K_i - N_i(\tau) \le x_i\} \quad ,$$

or

$$\theta(\underline{\lambda}, \underline{v}) = \mathrm{P}\{N_i(\tau) \ge K_i - x_i\} \quad .$$

Using the normal approximation which is applicable when the system is in heavy traffic

$$\begin{aligned}
\theta(\underline{\lambda}, \underline{v}) &= \mathrm{P}\{a\, m_i(\tau) + \sqrt{a}\, X_i(\tau) \ge K_i - x_i\} \\
&= \mathrm{P}\{\sqrt{a}\, X_i(\tau) \ge K_i - x_i - a\, m_i(\tau)\} \\
&= \mathrm{P}\left\{ X_i(\tau) \ge \frac{K_i - x_i - a\, m_i(\tau)}{\sqrt{a}} \right\} \\
&= \mathrm{P}\left\{ \frac{X_i(\tau)}{\sigma_i^2(\tau)} \ge \frac{K_i - x_i - a\, m_i(\tau)}{\sqrt{a\, \sigma_i^2(\tau)}} \right\} \\
&= 1 - \Phi\left(\frac{K_i - x_i - a\, m_i(\tau)}{\sqrt{a\, \sigma_i^2(\tau)}} \right) \quad ;
\end{aligned}$$

where Φ is the standard normal cumulative distribution function. In the foregoing it has been left implicit that $m_i(\tau) = m_i(\tau; \underset{\sim}{\lambda}, \underset{\sim}{\nu})$, and $\sigma_i^2(\tau) = \sigma_i^2(\tau; \underset{\sim}{\lambda}, \underset{\sim}{\nu})$. The MOE, $\theta(\underset{\sim}{\lambda}, \underset{\sim}{\nu})$, can be viewed as the conditional probability that the availability of item i is less than x_i at time τ. To remove the condition on $\underset{\sim}{\lambda}$ and $\underset{\sim}{\nu}$, leaving only the condition on the data, the bootstrap is used.

The Parametric Bayesian Bootstrapping Method.

a. Sample from gamma density (5.44) for λ_i, $i = 1, \ldots, I$, and from (5.45) for ν_i, $i = 1, \ldots, I$.

b. Compute $m_i(\tau)$, and $\sigma_i^2(\tau)$ using the diffusion approximation.

c. Calculate $\theta(\underset{\sim}{\lambda}, \underset{\sim}{\nu})$ using the standard normal distribution.

d. Repeat a. through c., say, 100 times and average the results.

Example 5.4: This example illustrates the use of the diffusion approximation in a Bayesian Bootstrapping application. It uses the dynamic-service-selection model with the probabilistic-longest-line (PLL;1) service discipline in the diffusion approximation, as an approximation to first-come-first-served (FCFS). The measure of effectiveness (MOE) to be examined in this example will be the probability that the number of each item remaining in operation drops to less than 50 at time 100, $P(K_i - X_i(100) \leq 50)$. The results for the following three cases are presented:

a. Case BB. Bayesian Bootstrapping with 100 replications.

b. Case AR. Using Average Rates calculated from the data, for failures and service completions, with the diffusion approximation to obtain a point estimate of the MOE. No Bayesian Bootstrapping.

c. Case TR. Using *True* Rates for failures and service completions with the diffusion approximation to obtain a point estimate of the MOE for comparison with the results obtained using limited data. These are the ***unknown*** true population parameters which were used to generate the failure time and service time data used in the example.

The inputs for this example are shown in Figure 26a. through c.. The data were generated by drawing from exponential distributions with parameters equal to the true rates.

i	1	2	3	4	5
K_i	135	115	255	165	135
ν_i	1.00	1.00	1.00	1.00	1.00
$N(0)$	0	0	0	0	0

MOE: $P(K_i - N_i(100) \leq 50)$

Figure 26a. Example 5.4 Inputs

Item	1	2	3	4	5
Number of failures	8	9	10	9	10
Data: time between failures	75	62	57	93	55
	22	170	20	34	4
	28	29	3	23	17
	183	77	11	101	239
	129	290	29	180	34
	6	71	56	57	11
	42	94	12	47	94
	69	95	17	8	52
		29	63	1	89
			45		76
Average failure rate	0.014	0.010	0.032	0.017	0.015
Given failure rate	0.010	0.020	0.030	0.020	0.010

Figure 26b. Example 5.4 Data: Times Between Failures

Item	1	2	3	4	5
Number of repairs	8	9	10	9	9
Data: times to repair	0.1	2.0	0.5	0.2	0.1
	4.1	0.4	0.1	0.1	0.3
	1.4	0.9	0.5	0.8	0.1
	1.7	1.9	0.1	0.2	0.4
	0.2	2.3	0.6	0.6	0.2
	2.7	2.0	0.1	0.1	0.2
	1.5	0.2	0.1	0.1	0.3
	0.7	0.1	0.4	0.1	0.1
		0.5	0.1	0.1	0.1
			0.2		
Average service rate	0.65	0.87	3.70	3.91	5.00
Given service rate	0.50	1.00	3.00	4.00	4.50

Figure 26c. Example 5.4 Data: Times to Repair

The results obtained for each case are shown in Figure 27a. and b.. Next to the resulting point estimates of the MOE under Case BB, in parentheses, are the standard errors obtained from the bootstrap.

Item	Case TR	Case AR	Case BB
	Expected Values		
1	61.5	46.6	47.5
2	26.6	54.4	55.9
3	34.0	33.8	37.5
4	38.1	49.7	51.4
5	61.5	45.1	46.4
	Standard Deviations		
1	5.9	5.7	5.5
2	4.7	5.5	5.4
3	6.5	6.6	6.7
4	6.0	6.4	6.4
5	6.1	5.9	5.8

Figure 27a. Example 5.4: Number in Operation at $t = 100$

Item	Case TR	Case AR	Case BB	
1	0.026	0.727	0.603	(0.401)
2	0.999	0.212	0.331	(0.373)
3	0.993	0.993	0.777	(0.329)
4	0.977	0.519	0.482	(0.415)
5	0.029	0.799	0.610	(0.406)

Figure 27b. Example 5.4 MOE: $P(K_i - N_i(100) \leq 50)$

Discussion of the Results: As should be anticipated, the results in each of the cases is different, the greatest differences occurring between the case that is based on the true population parameters, Case TR, and either of the cases that use the data, Cases AR and BB. But some significant differences also occurring between the two cases that use the data. The different results are most striking in Figure 27b. All the probabilities under case TR (true rates) are near 0 or 1, in contrast to the other cases. The greatest difference in the table is for item 2 under Cases TR and AR. The Bayesian Bootstrapping standard errors indicate that there is a very significant spread in the MOE due to the uncertainty in the underlying failure and service rates. A conclusion that may be reached from this small example is that with so little data and a non-informative prior, there is too much uncertainty in the rates to conclude that there are significant differences between items using this MOE.

F. GENERAL SERVICE DISTRIBUTIONS

Due to the generality of the renewal theory approach used in deriving the dynamic-service-selection correction to processor-sharing, that model is easily extended by relaxing the original assumption of exponential times to repair.

Start again with Equation (5.30), however now rather than assuming that the service times, S_i are exponentially distributed, simply retain the general form for the expected values, $E[S_i]$, and the second moments, $E[S_i^2]$, $i = 1, \dots, I$.

Completing the derivation for the expected cycle length gives the following:

$$E[C_i]^{-1} = \frac{\tilde{q}_i(\mathbf{n})}{E[S_i]} \quad , \tag{5.46}$$

where $\tilde{q}_i(\mathbf{n})$ has the general form specified in Proposition 5.2:

$$\tilde{q}_i(\mathbf{n}) = \frac{\tilde{w}_i f_i(n_i)}{\sum_k \tilde{w}_k f_k(n_k)} \quad ,$$

where $f_i(n_i)$ is an arbitrary function of n_i, but now $\tilde{w}_i$ is defined by $\tilde{w}_i = \mathrm{E}[S_i]\, w_i$.

Then, after deriving an expression for the second moment of the cycle length, the following expression analogous to (5.35) is obtained:

$$\frac{\mathrm{var}[C_i]}{\mathrm{E}[C_i]^3} = \frac{\tilde{q}_i(\mathbf{n})}{\mathrm{E}[S_i]}\left[1 + \tilde{q}_i(\mathbf{n})\left[\frac{\sum_j q_j(\mathbf{n})\,(\mathrm{var}[S_j] + \mathrm{E}[S_j]^2)}{\mathrm{E}[S_i]\sum_j q_j(\mathbf{n})\,\mathrm{E}[S_j]} - 2\right]\right] . \qquad (5.47)$$

Note that the $q_j(\mathbf{n})$ within both summations use the original weights w_j and not the modified weights $\tilde{w}_j = \mathrm{E}[S_j]\, w_j$, as are used in the $\tilde{q}_i(\mathbf{n})$. The simplification obtained in the case of exponential service times does not occur in the general service time distribution case.

Using (5.46) and (5.47), the model given by (5.29) becomes

$$dX_i(t) = \lambda_i\,(K_i - X_i(t))\,dt - \frac{\tilde{q}_i(\mathbf{n})}{\mathrm{E}[S_i]}\,dt$$
$$+ \left[\lambda_i(K_i - X_i(t)) + \frac{\tilde{q}_i(\mathbf{n})}{\mathrm{E}[S_i]}\left[1 + \tilde{q}_i(\mathbf{n})\left[\frac{\sum_j q_j(\mathbf{n})(\mathrm{var}[S_j] + \mathrm{E}[S_j]^2)}{\mathrm{E}[S_i]\sum_j q_j(\mathbf{n})\,\mathrm{E}[S_j]} - 2\right]\right]\right]^{\frac{1}{2}} dW_i(t) \quad , \qquad (5.48)$$

for $i = 1, \ldots, I$.

Previously, to apply the diffusion approximation, it was necessary to scale the original exponential service rates, v_i, and use $\mu_i = v_i/a$. Similar scaling is required for general service times. Let $\mu_{S_i} = a\,\mathrm{E}[S_i]$ and $\sigma^2_{S_i} = a^2\,\mathrm{var}[S_i]$. Note here that the symbol μ_S is being used as a scaled *mean service time* as opposed to the previous use of μ as a scaled exponential service *rate*. As before, let $K_i = a\,\alpha_i$, and assume a service selection probability,

$q_i(\mathbf{N}(t))$, of the form given in Proposition 5.1. Applying the diffusion approximation to (5.48), the following differential equations are obtained for the scaled mean queue lengths and covariance matrix elements:

$$dm_i(t) = \lambda_i (\alpha_i - m_i(t))\, dt - \frac{\tilde{q}_i(\mathbf{m}(t))}{\mu_{S_i}}\, dt \quad ; \tag{5.49}$$

for $i = 1, \dots, I$;

$$\frac{d\sigma_{ii}(t)}{dt} = (B_{ii}(t))^2 + 2\sum_{j=1}^{I}(H_{ij}(t)\, \sigma_{ij}(t)) \quad ; \tag{5.50}$$

and

$$\frac{d\sigma_{ij}(t)}{dt} = \sum_{k=1}^{I}[(H_{ik}(t)\, \sigma_{jk}(t)) + (H_{jk}(t)\, \sigma_{ik}(t))] \quad ; \tag{5.51}$$

for $i \neq j$; where

$$H_{ii}(t) = -\lambda_i - \frac{1}{\mu_{S_i}} \frac{\gamma\, c_i}{\beta_i + c_i\, m_i(t)}\, \tilde{q}_i(\mathbf{m}(t))\, (1 - \tilde{q}_i(\mathbf{m}(t))) \quad ,$$

$$H_{ij}(t) = \frac{1}{\mu_{S_i}} \frac{\gamma\, c_j}{\beta_j + c_j\, m_j(t)}\, \tilde{q}_i(\mathbf{m}(t))\, \tilde{q}_j(\mathbf{m}(t)) \quad ,$$

for $i \neq j$,

$$B_{ii}(t)^2 = \lambda_i\, (\alpha_i - m_i(t)) + V_i \quad ,$$

$$V_i = \frac{\tilde{q}_i(\mathbf{m}(t))}{\mu_{S_i}} \left[1 + \tilde{q}_i(\mathbf{m}(t)) \left[\frac{\sum_j q_j(\mathbf{m}(t))\, (\sigma_{Sj}^2 + \mu_{Sj}^2)}{\mu_{S_i} \sum_j q_j(\mathbf{m}(t))\, \mu_{Sj}} - 2 \right] \right] ,$$

and where

$$\tilde{q}_i(\mathbf{m}(t)) = \frac{w_i\,\mu_{S_i}\,[\beta_i + c_i\,m_i(t)]^\gamma}{\sum_j w_j\,\mu_{S_j}\,[\beta_j + c_j\,m_j(t)]^\gamma} \quad ,$$

and

$$q_i(\mathbf{m}(t)) = \frac{w_i\,[\beta_i + c_i\,m_i(t)]^\gamma}{\sum_j w_j\,[\beta_j + c_j\,m_j(t)]^\gamma} \quad .$$

Examples: Several examples are presented to demonstrate the results of using the diffusion approximation with general service times. The numerical solution results using the diffusion approximation are compared to corresponding simulation results. All of these examples use the PLL;1 service discipline. Example 5.5 shows the results when the service times are deterministic, and Example 5.6 shows the results when the service times are taken from a gamma distribution. For comparison, Examples 5.5 and 5.6 use the same mean service times and other common inputs, except, of course, for service time variance. Also, for comparison with the deterministic service times of Example 5.5, the gamma service times of Example 5.6 are taken to have a low coefficient of variation (i.e., variance one-tenth the variance of an exponential with the same mean). For higher coefficients of variation, Example 5.7 examines cases in which service times are taken from a log-normal distribution and a gamma distribution with variances four times the variance of an exponential with the same mean. Example 5.7 also examines the effect of varying the weights, w_i, in the service discipline function. Specifically, equal weights are compared with weights set to the item traffic intensity $\rho_i = \lambda_i\,E[S_i]$.

Example 5.5: This example demonstrates the results of using the diffusion approximation when the service times are deterministic. The numerical solution results are compared to corresponding simulation results. This example uses the PLL;1 service discipline. The inputs for this example are shown in Figure 28.

i	1	2	3	4	5
K_i	50	100	150	200	250
λ_i	.01	.02	.03	.02	.01
w_i	1.0	1.0	1.0	1.0	1.0
$N(0)$	0	0	0	0	0
$E(S_i)$	2.0	1.0	.3333	.25	.2222
$var(S_i)$	0	0	0	0	0

Figure 28. Example 5.5 Inputs

Transient Results: For Example 5.5, the transient response of the system is summarized in tabular form in Figure 29, and then one of the queues is displayed graphically as a function of time. The tabulated results, showing the mean and standard deviation of the number in each queue as a function of time, are given at selected times. The plot in Figure 30 show the solutions for both the mean queue length for that item and standard deviation of queue length to compare diffusion approximation results with corresponding simulation results. Mean queue length and standard deviations were computed at unit time steps. Upper and lower .95 confidence limits for the means are the point estimate $\pm .0877$ times the corresponding estimate for the standard deviation (i.e., about $\pm$ 10% of the standard deviation). Upper and lower .95 confidence limits for the standard deviations are .942 and 1.066 times the point estimate (i.e., about $\pm$ 5%). At all t, the results show that the diffusion approximation yields solutions close to the results from the simulation.

	Means:					Standard Deviations:				
Item:	1	2	3	4	5	1	2	3	4	5
t = 50										
Diff.	16.1	52.9	99.8	105.9	80.4	3.4	5.3	6.6	8.0	8.0
Sim.	16.4	52.9	99.4	106.3	80.7	3.1	4.9	6.0	7.5	7.7
t = 100										
Diff.	25.3	72.6	124.4	145.3	126.4	3.6	4.6	5.1	7.2	8.8
Sim.	25.4	72.7	123.9	144.9	126.0	3.4	4.4	4.8	6.9	8.7
t = 150										
Diff.	30.5	80.1	130.9	160.2	152.5	3.5	3.9	4.4	6.3	8.7
Sim.	30.6	80.1	130.4	159.9	152.3	3.4	3.8	4.3	6.0	7.9
t = 200										
Diff.	33.4	83.0	133.0	166.0	167.2	3.3	3.6	4.1	5.9	8.5
Sim.	33.4	83.1	132.6	165.6	166.7	3.1	3.7	4.2	5.9	8.1
t = 250										
Diff.	35.1	84.2	133.8	168.4	175.3	3.1	3.5	4.1	5.7	8.3
Sim.	34.9	84.2	133.5	168.0	174.6	3.1	3.4	4.0	5.3	7.7
t = 300										
Diff.	36.0	84.7	134.1	169.4	179.8	3.0	3.4	4.0	5.6	8.2
Sim.	36.0	84.3	133.7	168.9	179.2	3.2	3.5	3.9	5.7	7.5
t = 350										
Diff.	36.4	84.9	134.2	169.9	182.2	3.0	3.4	3.9	5.6	8.1
Sim.	36.7	84.8	133.9	169.6	181.8	2.8	3.6	3.8	5.7	7.6
t = 400										
Diff.	36.7	85.1	134.3	170.1	183.7	3.0	3.4	3.9	5.5	8.0
Sim.	37.1	84.7	133.9	169.8	183.5	2.6	3.3	4.0	5.5	7.5

Figure 29. Example 5.5: Transient $N_i(t)$

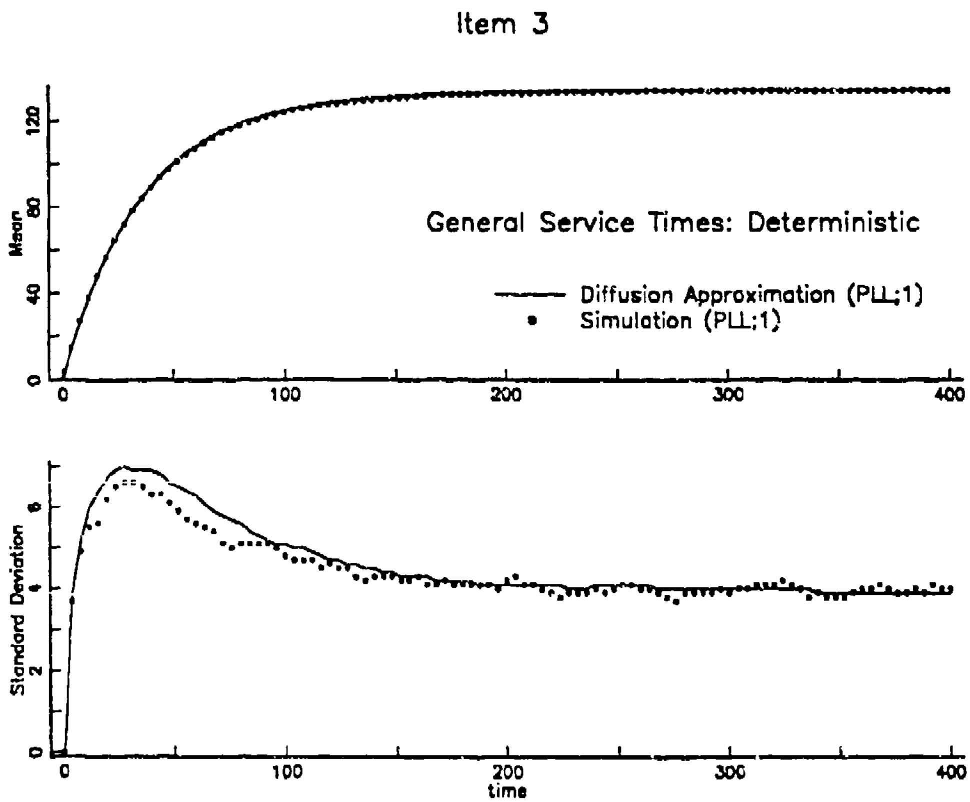


Figure 30. Example 5.5: $N_3(t)$

Example 5.6: This example demonstrates the results of using the diffusion approximation when the service times come from a gamma distribution with a high shape parameter, and consequently low variance compared to an exponential with the same mean. The numerical solution results are compared to corresponding simulation results. This example uses the PLL;1 service discipline. Except for service time variance, the inputs for this example are the same as for Example 5.5 and are shown in Figure 31.

i	1	2	3	4	5
K_i	50	100	150	200	250
λ_i	.01	.02	.03	.02	.01
w_i	1.0	1.0	1.0	1.0	1.0
$N(0)$	0	0	0	0	0
$E(S_i)$	2.0	1.0	.3333	.25	.2222
$var(S_i)$	.40	.10	.011111	.0625	.004938

Figure 31. Example 5.6 Inputs

Transient Results: For Example 5.6, the transient response of the system is summarized in tabular form in Figure 32, and then one of the queues is displayed graphically as a function of time. The tabulated results, showing the mean and standard deviation of the number in each queue as a function of time, are given at selected times. The plot in Figure 33 show the solutions for both the mean queue length for that item and standard deviation of queue length to compare diffusion approximation results with corresponding simulation results. Mean queue length and standard deviations were computed at unit time steps. Upper and lower .95 confidence limits for the means are the point estimate $\pm .0877$ times the corresponding estimate for the standard deviation (i.e., about $\pm$ 10% of the standard deviation). Upper and lower .95 confidence limits for the standard deviations are .942 and 1.066 times the point estimate (i.e., about $\pm$ 5%). At all t, the results show that the diffusion approximation yields solutions close to the results from the simulation.

	Means:					Standard Deviations:				
Item:	1	2	3	4	5	1	2	3	4	5
t = 50										
Diff.	16.1	52.9	99.8	105.9	80.4	3.4	5.3	6.6	8.1	8.1
Sim.	16.4	52.7	99.8	105.7	80.3	3.5	4.6	6.0	7.6	7.8
t = 100										
Diff.	25.3	72.6	124.4	145.3	126.4	3.6	4.6	5.1	7.2	8.8
Sim.	25.4	72.5	124.1	145.3	126.3	3.5	4.3	4.6	6.3	8.2
t = 150										
Diff.	30.5	80.1	130.9	160.2	152.5	3.5	4.0	4.4	6.4	8.7
Sim.	30.3	80.0	130.7	159.9	152.3	3.4	4.0	4.3	6.1	8.1
t = 200										
Diff.	33.4	83.0	133.0	166.0	167.2	3.3	3.7	4.2	6.0	8.5
Sim.	33.3	82.7	132.7	165.8	167.5	3.4	3.4	4.1	5.8	8.4
t = 250										
Diff.	35.1	84.2	133.8	168.4	175.3	3.1	3.5	4.2	5.8	8.4
Sim.	35.0	84.1	133.6	168.2	175.2	2.9	3.6	4.1	5.8	8.1
t = 300										
Diff.	36.0	84.7	134.1	169.4	179.8	3.1	3.5	4.0	5.7	8.2
Sim.	35.7	84.8	133.8	169.1	180.2	3.1	3.4	4.2	5.3	8.2
t = 350										
Diff.	36.4	84.9	134.2	169.9	182.2	3.0	3.4	4.0	5.6	8.2
Sim.	36.3	85.0	133.9	169.4	182.4	3.0	3.3	4.0	5.6	7.6
t = 400										
Diff.	36.7	85.1	134.3	170.1	183.7	3.0	3.4	4.0	5.6	8.1
Sim.	36.6	85.0	134.2	169.9	183.4	2.9	3.4	4.0	5.5	7.5

Figure 32. **Example 5.6: Transient $N_i(t)$**

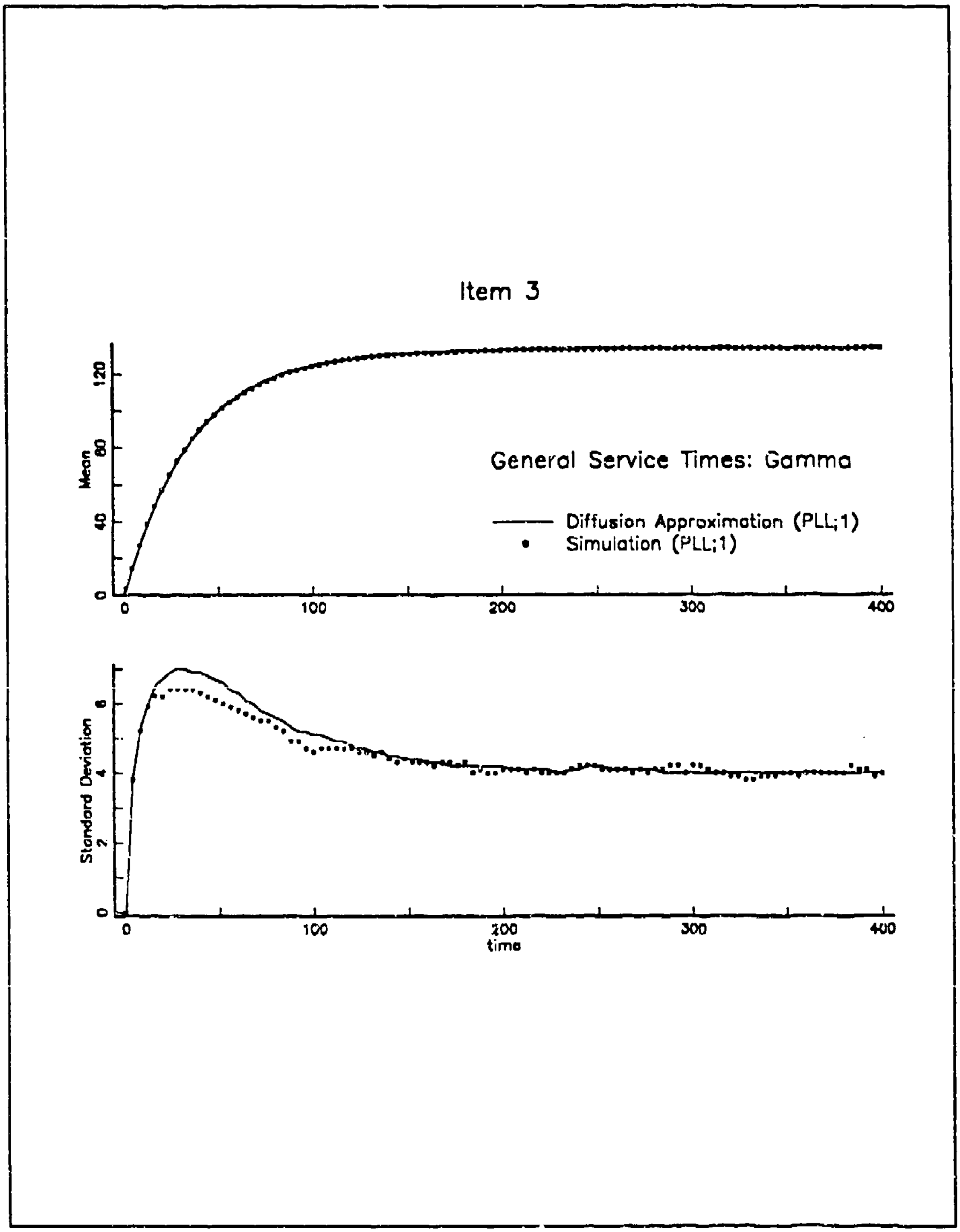


Figure 33. Example 5.6: $N_3(t)$

Example 5.7: This example demonstrates the results of using the diffusion approximation when the service times come from a distribution with a higher coefficient of variation. In this example, the variance of the service times are four times the variance of an exponential with the same mean, i.e., twice the standard deviation of the exponential. The simulations use service times taken from a log-normal distribution and a gamma distribution. In this example, the probabilistic-longest-line service discipline, PLL;1, is used. But in addition to unit weights, $w_i = 1$, the effect of varying the weights is examined. Specifically unit weights are compared with weighting the queue lengths by their respective traffic intensities $\rho_i = \lambda_i \mathrm{E}[S_i]$. In addition to comparing the diffusion approximation solution with simulations for service time coefficients of variation of 2, the diffusion approximation solutions for exponential service times (coefficient of variation of 1) and deterministic service times (coefficient of variation of 0) are also presented.

The ten cases presented in this example are identified by the following codes:

A; 1; 0	A; ρ; 0
A; 1; 1	A; ρ; 1
A; 1; 2	A; ρ; 2
G; 1; 2	G; ρ; 2
L; 1; 2	L; ρ; 2

The first position indicates the solution method. Here the letter A represents the diffusion approximation solution, the letter G, the simulation with the gamma service time distribution, and the letter L, the simulation with log-normal service times. The second position indicates the type of weights. Here the number 1 represents equal unit weights, and the letter ρ represents traffic intensity weights. The third position indicates the coefficient of variation. Here zero represents deterministic service times, 1 represents exponential service times, and 2 represents general service times. The inputs for this example are shown in Figure 34.

i	1	2	3	4	5
K_i	100	110	120	130	140
λ_i	.0110	.0130	.0150	.0160	.0170
$E(S_i)$	0.50	0.40	0.30	0.25	0.20
$N(0)$	0	0	0	0	0
$var(S_i)(0)$	0.0	0.0	0.0	0.0	0.0
$var(S_i)(1)$	0.25	0.16	0.09	0.0625	0.04
$var(S_i)(2)$	1.00	0.64	0.36	0.25	0.16
$w_i(1)$	1.00	1.00	1.00	1.00	1.00
$w_i(\rho)$	0.0055	0.0052	0.0045	0.0040	0.0034

Figure 34. Example 5.7 Inputs

Transient Results: For Example 5.7, the transient response of the system is summarized in tabular form in Figure 35a. and b., and then one of the queues is displayed graphically as a function of time. The tabulated results, showing the mean and standard deviation of the number in each queue as a function of time, are given at selected times. The plots in Figure 36 show the solutions for both the mean queue length for that item and standard deviation of queue length to compare diffusion approximation results with corresponding simulation results. Mean queue lengths and standard deviations were computed at unit time steps. Upper and lower .95 confidence limits for the means are the point estimate $\pm$.0877 times the corresponding estimate for the standard deviation (i.e., about $\pm$ 10% of the standard deviation). Upper and lower .95 confidence limits for the standard deviations are .942 and 1.066 times the point estimate (i.e., about $\pm$ 5%).

Discussion of the Tabulated Results: At all t, the results show that the diffusion approximation yields solutions close to the results from the simulation. Within the grouping of results for by type of weights, it is seen that there is very close agreement in the means for all solution methods and all coefficients of variation. For the standard deviations, there is a clear pattern of systematic differences due to the coefficient of variation and the service time distribution used in the simulation. Part of the systematic differences are anticipated. It is quite reasonable to expect that as the coefficient of variation of the service times changes from 0 to 1 to 2, that the variation in the queue lengths also increases. That is reflected in the table. The other differences, between analytic and simulation solutions with the coefficient of variation of 2, show that there

are certainly higher moment effects that are not fully captured by the diffusion approximation.

Item:	Means: 1	2	3	4	5	Standard Deviations: 1	2	3	4	5
	t = 50									
A; 1; 0	26.0	32.7	39.8	45.3	51.0	4.79	5.25	5.71	6.09	6.51
A; 1; 1	26.0	32.7	39.8	45.3	51.0	4.92	5.43	5.93	6.36	6.81
A; 1; 2	25.9	32.6	39.7	45.1	50.8	5.27	5.89	6.52	7.05	7.63
G; 1; 2	25.8	32.6	39.4	44.9	50.2	5.11	5.88	6.54	7.21	7.47
L; 1; 2	25.2	32.0	38.8	44.1	50.2	5.17	6.29	6.80	8.02	8.14
A; ρ; 0	24.0	30.9	39.9	47.3	56.1	4.74	5.25	5.78	6.19	6.58
A; ρ; 1	23.9	30.8	39.7	47.1	55.8	4.88	5.43	5.99	6.41	6.80
A; ρ; 2	23.6	30.3	39.1	46.4	54.9	5.27	5.93	6.56	7.02	7.40
G; ρ; 2	23.8	30.9	40.2	46.6	55.2	5.36	6.46	6.84	7.28	7.48
L; ρ; 2	23.9	30.7	39.3	46.6	55.3	5.38	5.83	7.08	7.48	8.34
	t = 100									
A; 1; 0	40.2	49.3	58.8	66.1	73.7	5.23	5.57	5.91	6.23	6.58
A; 1; 1	40.2	49.3	58.8	66.1	73.7	5.40	5.79	6.16	6.51	6.88
A; 1; 2	40.1	49.3	58.7	66.0	73.6	5.88	6.40	6.91	7.38	7.87
G; 1; 2	40.3	49.0	58.6	65.7	73.4	5.81	6.55	6.81	7.36	7.92
L; 1; 2	40.1	48.7	58.3	65.6	73.1	6.04	6.63	7.53	7.79	8.60
A; ρ; 0	37.0	46.7	58.9	69.1	80.9	5.21	5.61	5.97	6.25	6.46
A; ρ; 1	36.9	46.6	58.8	69.0	80.8	5.41	5.85	6.24	6.53	6.74
A; ρ; 2	36.7	46.4	58.6	68.6	80.4	5.97	6.54	6.98	7.30	7.49
G; ρ; 2	36.7	46.4	58.8	68.6	80.3	6.11	6.68	7.50	7.79	7.84
L; ρ; 2	36.9	46.6	58.4	68.7	80.6	6.01	6.84	7.41	7.73	8.12

Figure 35a. Example 5.7: Transient $N_i(t)$

Item:	Means: 1	2	3	4	5	Standard Deviations: 1	2	3	4	5
	t = 150									
A; 1; 0	47.8	57.7	67.8	75.8	83.9	5.20	5.46	5.72	5.99	6.27
A; 1; 1	47.8	57.8	67.9	75.9	84.1	5.40	5.71	6.01	6.32	6.64
A; 1; 2	47.8	57.8	67.9	75.8	84.0	5.96	6.39	6.82	7.24	7.67
G; 1; 2	48.0	57.5	67.4	[illegible]	83.9	5.93	6.51	7.65	7.96	7.70
L; 1; 2	48.1	57.4	67.6	[illegible]	83.6	6.18	6.79	7.24	7.83	8.33
A; ρ; 0	44.0	54.7	68.1	79.2	92.1	5.22	5.53	5.76	5.95	6.05
A; ρ; 1	43.9	54.7	68.1	79.2	92.0	5.46	5.82	6.08	6.27	6.36
A; ρ; 2	43.5	54.2	67.5	73.6	91.3	6.09	6.58	6.89	7.12	7.18
G; ρ; 2	43.8	54.5	67.7	78.9	92.0	5.83	7.09	7.01	7.68	7.50
L; ρ; 2	43.3	54.5	67.5	78.8	91.4	6.07	6.93	7.19	7.88	8.12
	t = 200									
A; 1; 0	51.7	61.8	72.0	80.2	88.5	5.12	5.34	5.57	5.82	6.08
A; 1; 1	51.9	62.1	72.3	80.5	88.9	5.33	5.60	5.87	6.16	6.46
A; 1; 2	51.9	62.1	72.3	80.5	88.8	5.93	6.33	6.71	7.10	7.49
G; 1; 2	51.7	62.1	71.6	80.3	88.5	6.36	6.38	7.01	7.48	7.98
L; 1; 2	51.7	61.7	71.9	79.9	88.5	6.40	6.59	7.30	7.62	8.78
A; ρ; 0	47.7	58.8	72.5	84.0	97.2	5.16	5.43	5.62	5.76	5.80
A; ρ; 1	47.5	58.5	72.3	83.7	96.9	5.42	5.73	5.93	6.08	6.11
A; ρ; 2	47.2	58.2	72.0	83.4	96.5	6.11	6.55	6.79	6.96	6.96
G; ρ; 2	47.5	58.4	71.9	83.6	96.8	6.17	6.99	7.19	7.43	7.58
L; ρ; 2	47.0	58.2	71.9	83.3	96.5	6.59	7.15	7.23	7.79	7.88
	t = 250									
A; 1; 0	53.8	64.0	74.2	82.4	90.8	5.06	5.27	5.48	5.72	5.97
A; 1; 1	54.0	64.2	74.4	82.7	91.1	5.28	5.53	5.79	6.07	6.35
A; 1; 2	54.1	64.3	74.5	82.7	91.1	5.91	6.28	6.64	7.02	7.42
G; 1; 2	53.9	64.2	74.1	83.0	91.0	5.66	6.61	6.57	6.92	7.94
L; 1; 2	54.1	64.1	74.6	82.6	90.5	6.41	7.00	7.18	7.99	8.26
A; ρ; 0	49.6	60.8	74.7	86.4	99.6	5.11	5.35	5.50	5.62	5.63
A; ρ; 1	49.4	60.6	74.5	86.1	99.3	5.38	5.67	5.84	5.97	5.97
A; ρ; 2	49.4	60.6	74.4	86.0	99.3	6.11	6.51	6.71	6.86	6.82
G; ρ; 2	49.8	61.1	74.3	86.0	99.4	6.18	6.72	6.99	7.26	7.50
L; ρ; 2	49.3	60.2	74.3	85.4	99.0	6.40	6.88	7.40	7.50	7.10

Figure 35b. Example 5.7: Transient $N_i(t)$ (cont.)

Item:	Means: 1	2	3	4	5	Standard Deviations: 1	2	3	4	5
	t = 300									
A; 1; 0	55.0	65.2	75.3	83.6	92.0	5.02	5.22	5.43	5.67	5.91
A; 1; 1	55.1	65.3	75.4	83.7	92.2	5.25	5.49	5.75	6.02	6.30
A; 1; 2	55.2	65.4	75.5	83.8	92.3	5.89	6.25	6.60	6.97	7.33
G; 1; 2	54.9	65.4	75.7	84.1	92.4	5.87	6.37	6.69	7.14	7.80
L; 1; 2	55.6	65.2	75.7	84.0	92.0	6.34	6.91	7.36	7.92	8.02
A; ρ; 0	50.6	61.9	75.8	87.5	100.8	5.08	5.32	5.47	5.58	5.58
A; ρ; 1	50.4	61.7	75.6	87.3	100.5	5.36	5.64	5.79	5.91	5.90
A; ρ; 2	50.5	61.7	75.6	87.3	100.6	6.10	6.49	6.67	6.80	6.75
G; ρ; 2	50.5	62.8	75.7	87.9	101.2	6.02	6.79	6.63	6.91	7.25
L; ρ; 2	50.6	61.9	75.6	86.7	100.5	6.46	7.38	7.11	7.65	7.64
	t = 350									
A; 1; 0	55.7	65.8	75.9	84.2	92.7	4.99	5.19	5.41	5.64	5.88
A; 1; 1	55.7	65.9	76.0	84.3	92.7	5.23	5.47	5.72	5.99	6.27
A; 1; 2	55.8	65.9	76.1	84.4	92.8	5.88	6.23	6.58	6.95	7.31
G; 1; 2	55.7	65.8	76.0	84.7	92.9	6.11	6.31	6.83	6.78	7.53
L; 1; 2	55.8	65.9	75.7	84.3	92.5	6.47	6.22	7.48	7.35	7.95
A; ρ; 0	51.1	62.4	76.3	88.0	101.3	5.06	5.29	5.42	5.53	5.53
A; ρ; 1	51.0	62.3	76.2	87.9	101.2	5.34	5.62	5.76	5.87	5.85
A; ρ; 2	51.0	62.3	76.2	87.9	101.2	6.10	6.48	6.64	6.77	6.71
G; ρ; 2	51.1	62.9	76.4	88.3	101.5	6.23	6.99	6.77	6.82	7.08
L; ρ; 2	50.8	62.0	75.6	87.1	100.7	6.52	7.31	7.28	7.82	7.22
	t = 400									
A; 1; 0	56.0	66.1	76.2	84.5	93.0	4.98	5.18	5.39	5.62	5.87
A; 1; 1	56.0	66.2	76.3	84.6	93.0	5.22	5.46	5.71	5.98	6.26
A; 1; 2	56.1	66.2	76.3	84.6	93.1	5.87	6.22	6.57	6.94	7.31
G; 1; 2	55.9	65.9	76.3	84.2	92.9	6.06	6.80	6.52	6.99	7.70
L; 1; 2	56.3	66.1	76.2	84.9	93.0	6.18	6.96	7.62	7.42	8.17
A; ρ; 0	51.4	62.7	76.6	88.3	101.6	5.05	5.28	5.42	5.53	5.52
A; ρ; 1	51.3	62.6	76.5	88.3	101.6	5.33	5.60	5.75	5.85	5.83
A; ρ; 2	51.3	62.6	76.5	88.2	101.5	6.10	6.47	6.63	6.75	6.69
G; ρ; 2	51.4	62.8	76.3	88.7	101.8	6.28	6.65	6.95	6.82	6.99
L; ρ; 2	51.1	62.8	76.3	87.1	101.1	6.62	6.96	7.66	7.65	7.10

Figure 35c. Example 5.7: Transient $N_i(t)$ (cont.)

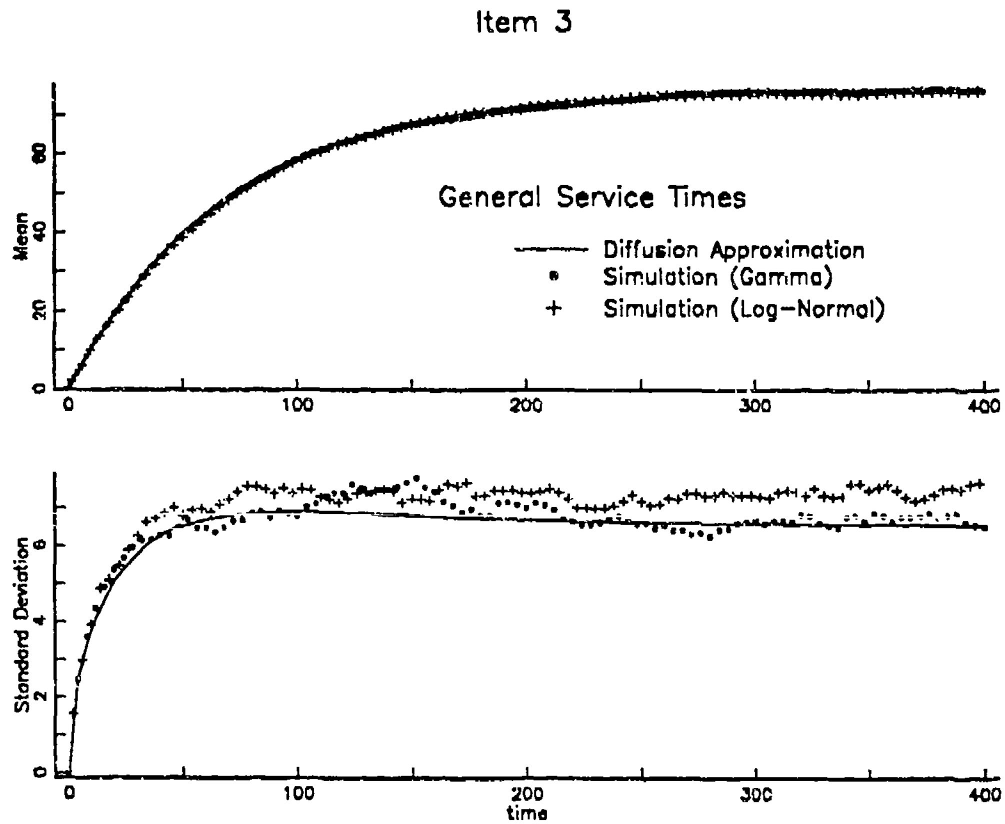


Figure 36a. Example 5.7: $N_3(t)$; Unit Weights

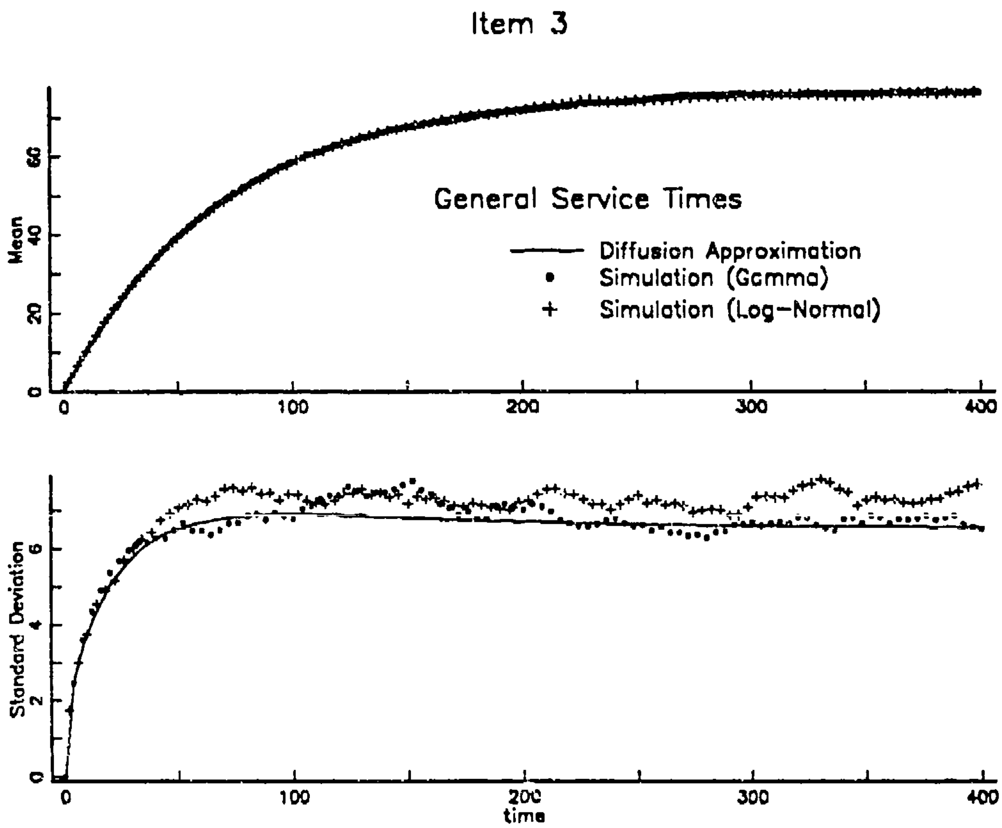


Figure 36b. Example 5.7: $N_3(t)$; Traffic Intensity Weights

G. THE AIRCRAFT DETACHMENT REPAIRMAN MODEL

This is a model of the aircraft detachment repairman problem with queue length influence. As before, the arrival rate of each type of failed item as seen at the maintenance shop will be the individual item failure rate, λ_i, multiplied by the number of items operating at that time. However, in this model, the number of items operating is limited by the number of operational aircraft available.

Let A_c denote the number of aircraft assigned to the detachment. And let $A_v(t)$ denote the number of aircraft that are operationally available at time t, defined by

$$A_v(t) = \min \{A_c, K_1 - N_1(t), K_2 - N_2(t), \dots, K_I - N_I(t)\} \quad . \tag{5.52}$$

Hence, the time-dependent arrival rate of each type of failed item as seen by the repairman is $\lambda_i A_v(t)$, for $i = 1, \dots, I$, and the probability that a failed item of type i arrives at the repair shop in the interval $(t\,,\ t + dt)$ is $\lambda_i A_v(t)\, dt + o(dt)$.

This modified arrival process, which will be carried through the derivation of the diffusion approximation, is the only new aspect of this model. Any of the previously considered service processes and disciplines could be used here. Arbitrarily, the service times will be assumed to be exponential with mean $1/v_i$ and the dynamic-service-selection correction to processor-sharing will be used. Then, modifying only the failure process terms in Equation (5.36), the following system of stochastic differential equations are obtained:

$$\begin{aligned} dN_i(t) = {} & \lambda_i A_v(t)\, dt \ - \ v_i \tilde{q}_i(\mathbf{N}(t))\, dt \\ & + \sqrt{\lambda_i A_v(t) + v_i \tilde{q}_i(\mathbf{N}(t))\left(1 + 2\,\tilde{q}_i(\mathbf{N}(t))\left\{v_i \sum_j \frac{\tilde{q}_j(\mathbf{N}(t))}{v_j} - 1\right\}\right)}\; dW_i(t)\,. \end{aligned} \tag{5.53}$$

for $i = 1, \dots, I$; where $\{W_i(t);\, t \geq 0\}$ are independent standard Wiener processes.

To deal with the minimization operator within the differential equations, two cases must now be considered. Case I will be the case when the available number of operational aircraft is not limited by part availability (i.e., operational aircraft equal to aircraft assigned), and Case II will be the case when the available number of operational aircraft *is* limited by the availability of one of the parts.

Case I. If at some time t, $A_v(t) = A_c$, then, at t, (5.53) may be expressed as

$$dN_i(t) = \lambda_i A_c\, dt - v_i \tilde{q}_i(\mathbf{N}(t))\, dt + \sqrt{\lambda_i A_c + v_i \tilde{q}_i(\mathbf{N}(t))\left(1 + 2\tilde{q}_i(\mathbf{N}(t))\left\{v_i \sum_j \frac{\tilde{q}_j(\mathbf{N}(t))}{v_j} - 1\right\}\right)}\; dW_i(t)\,. \tag{5.54}$$

Now let $a = A_c + \sum_j K_j$, the total population of components and aircraft, and again consider the approximation

$$N_i(t) = a\, m_i(t) + \sqrt{a}\, X_i(t) \quad .$$

As before, let $v_i = \mu_i a$, and use the general service discipline form and expansion of Proposition 5.1. Now expressing A_c as a fraction of a, let $A_c = \alpha_c a$. After dividing through by a, (5.54) becomes

$$\begin{aligned} dm_i(t) + \frac{1}{\sqrt{a}} dX_i(t) &= \lambda_i \alpha_c\, dt \\ &- \mu_i \tilde{q}_i(\mathbf{m}(t))\left(1 + \frac{1}{\sqrt{a}}\left(X_i(t)\frac{\gamma c_i}{\beta_i + c_i m_i(t)} - \sum_j X_j(t)\frac{\gamma c_j}{\beta_j + c_j m_j(t)}\tilde{q}_j(\mathbf{m}(t))\right)\right)dt \\ &+ \frac{1}{\sqrt{a}}\left[\lambda_i \alpha_c + \mu_i \tilde{q}_i(\mathbf{m}(t))\left(1 + 2\tilde{q}_i(\mathbf{m}(t))\left\{\mu_i \sum_j \frac{\tilde{q}_j(\mathbf{m}(t))}{\mu_j} - 1\right\}\right)\right]^{1/2} dW_i(t) \\ &+ o(1/\sqrt{a}) \quad , \end{aligned}$$

where

$$\tilde{q}_i(\mathbf{m}(t)) = \frac{\frac{w_i}{\mu_i}\,[\beta_i + c_i m_i(t)]^\gamma}{\sum_j \frac{w_j}{\mu_j}\,[\beta_j + c_j m_j(t)]^\gamma} \quad .$$

Isolating terms of like order, the following systems of differential equations are obtained:

Equations of Order 1. The equations of order 1 form the following system of ordinary differential equations:

$$dm_i(t) = \lambda_i \alpha_c\, dt - \mu_i \tilde{q}_i(\mathbf{m}(t))\, dt \quad , \tag{5.55}$$

for $i = 1, \dots, I$.

Equations of Order $1/\sqrt{a}$. The equations of order $1/\sqrt{a}$ form the following system of stochastic differential equations:

$$dX_i(t) = -\mu_i \tilde{q}_i(\mathbf{m}(t)) \left(X_i(t) \frac{\gamma c_i}{\beta_i + c_i m_i(t)} - \sum_j X_j(t) \frac{\gamma c_j}{\beta_j + c_j m_j(t)} \tilde{q}_j(\mathbf{m}(t)) \right) dt$$
$$+ \sqrt{\lambda_i \alpha_c + \mu_i \tilde{q}_i(\mathbf{m}(t)) \left(1 + 2 \tilde{q}_i(\mathbf{m}(t)) \left\{ \mu_i \sum_j \frac{\tilde{q}_j(\mathbf{m}(t))}{\mu_j} - 1 \right\} \right)} \, dW_i(t) , \qquad (5.56)$$

for $i = 1, \ldots, I$.

Case II. If at some time t, $A_s(t) = K_s - N_s(t)$, where the subscript s denotes the part with the smallest availability, then, at t, (5.53) may be expressed as

$$dN_i(t) = \lambda_i K_s - N_s(t) \, dt - v_i \tilde{q}_i(\mathbf{N}(t)) \, dt$$
$$+ \sqrt{\lambda_i K_s - N_s(t) + v_i \tilde{q}_i(\mathbf{N}(t)) \left(1 + 2 \tilde{q}_i(\mathbf{N}(t)) \left\{ v_i \sum_j \frac{\tilde{q}_j(\mathbf{N}(t))}{v_j} - 1 \right\} \right)} \, dW_i(t) \qquad (5.57)$$

Using the same approximation, service discipline, and scaling of parameters as in Case I, and $K_s = \alpha_s a$, and then after dividing through by a, (5.57) becomes

$$dm_i(t) + \frac{1}{\sqrt{a}} dX_i(t) = \lambda_i \left(\alpha_s - m_s(t) - \frac{1}{\sqrt{a}} X_s(t) \right) dt$$
$$- \mu_i \tilde{q}_i(\mathbf{m}(t)) \left(1 + \frac{1}{\sqrt{a}} \left(X_i(t) \frac{\gamma c_i}{\beta_i + c_i m_i(t)} - \sum_j X_j(t) \frac{\gamma c_j}{\beta_j + c_j m_j(t)} \tilde{q}_j(\mathbf{m}(t)) \right) \right) dt$$
$$+ \frac{1}{\sqrt{a}} \left[\lambda_i \left(\alpha_s - m_s(t) - \frac{1}{\sqrt{a}} X_s(t) \right) \right.$$
$$\left. + \mu_i \tilde{q}_i(\mathbf{m}(t)) \left(1 + 2 \tilde{q}_i(\mathbf{m}(t)) \left\{ \mu_i \sum_j \frac{\tilde{q}_j(\mathbf{m}(t))}{\mu_j} - 1 \right\} \right) \right]^{1/2} dW_i(t) + o(1/\sqrt{a}) ,$$

where $\tilde{q}_i(\mathbf{m}(t))$ is as given in Case I. Isolating terms of like order, the following systems of differential equations are obtained:

Equations of Order 1. The equations of order 1 form the following system of ordinary differential equations:

$$dm_i(t) = \lambda_i \, (\alpha_s - m_s(t)) \, dt - \mu_i \tilde{q}_i(\mathbf{m}(t)) \, dt , \qquad (5.58)$$

for $i = 1, \ldots, I$.

Equations of Order $1/\sqrt{a}$. The equations of order $1/\sqrt{a}$ form the following system of stochastic differential equations:

$$dX_i(t) = -\lambda_i X_s(t)\,dt - \mu_i \tilde{q}_i(\mathbf{m}(t)) \left(X_i(t) \frac{\gamma c_i}{\beta_i + c_i m_i(t)} - \sum_j X_j(t) \frac{\gamma c_j}{\beta_j + c_j m_j(t)} \tilde{q}_j(\mathbf{m}(t)) \right) dt$$
$$+ \sqrt{\lambda_i (\alpha_s - m_s(t)) + \mu_i \tilde{q}_i(\mathbf{m}(t)) \left(1 + 2\tilde{q}_i(\mathbf{m}(t)) \left\{ \mu_i \sum_j \frac{\tilde{q}_j(\mathbf{m}(t))}{\mu_j} - 1 \right\} \right)}\; dW_i(t)\,, \quad (5.59)$$

for $i = 1, \ldots, I$.

The implication of the different cases to deal with the minimization operator is that as the controlling element of aircraft availability changes over time, the differential equations which approximately describe the behavior of the system change. Since the differential equations require numerical solution in any case, the solver just has to be able to distinguish which case applies at each time step in the solution. The case may be determined up to order a by the minimization

$$\min \left\{ \alpha_c\,,\ (\alpha_1 - m_1(t))\,,\ \ldots\,,\ (\alpha_I - m_I(t)) \right\} \ .$$

This is justified under the conditions previously specified for the use of the diffusion approximation, i.e., large populations from which the failed parts arrive, $a \to \infty$. Define the deterministic function $\alpha_v(t)$ by

$$\alpha_v(t) = \min \left\{ \alpha_c\,,\ (\alpha_1 - m_1(t))\,,\ \ldots\,,\ (\alpha_I - m_I(t)) \right\} \ .$$

Then the applicable case is determined as following:

$$\{ \alpha_v(t) = \alpha_c \} \quad \Rightarrow \quad \text{Case I} \ .$$

$$\{ \alpha_v(t) = (\alpha_s - m_s(t)) \} \quad \Rightarrow \quad \text{Case II} \ .$$

Using the function $\alpha_v(t)$ to imply the applicable case, Equations (5.55) and (5.58) reduce to the common form

$$dm_i(t) = \lambda_i\, \alpha_v(t)\, dt - \mu_i\, \tilde{q}_i(\mathbf{m}(t))\, dt \quad , \qquad (5.60)$$

for $i = 1, \dots, I$. Thus (5.60) is the system of ordinary differential equations for the time-dependent scaled mean queue lengths, for both Cases I and II. Similarly, for the stochastic differential equations, (5.56) and (5.59) reduce to the common form

$$
\begin{aligned}
dX_i(t) = & -\lambda_i X_s(t)\, \delta(\alpha_v(t), \alpha_s - m_s(t))\, dt \\
& - \mu_i\, \tilde{q}_i(\mathbf{m}(t)) \left(X_i(t) \frac{\gamma\, c_i}{\beta_i + c_i\, m_i(t)} - \sum_j X_j(t) \frac{\gamma\, c_j}{\beta_j + c_j\, m_j(t)}\, \tilde{q}_j(\mathbf{m}(t)) \right) dt \\
& + \sqrt{\lambda_i\, \alpha_v(t) + \mu_i\, \tilde{q}_i(\mathbf{m}(t)) \left(1 + 2\, \tilde{q}_i(\mathbf{m}(t)) \left\{ \mu_i \sum_j \frac{\tilde{q}_j(\mathbf{m}(t))}{\mu_j} - 1 \right\} \right)}\; dW_i(t)\,,
\end{aligned}
\tag{5.61}
$$

for $i = 1, \dots, I$; where $\delta(\alpha_v(t), \alpha_s - m_s(t))$ is used as the Kronecker delta.[5]

As before, (5.61) can be written in matrix form as

$$
d\mathbf{X}(t) = \mathbf{H}(t)\, \mathbf{X}(t)\, dt + \mathbf{B}(t)\, d\mathbf{W}(t) \quad ; \tag{5.62}
$$

where $\mathbf{H}(t)$ is the $I \times I$ matrix

$$
\mathbf{H}(t) = \mathbf{H}^1(t) - \mathbf{H}^2(t)\, \delta(\alpha_v(t), \alpha_s - m_s(t)) \quad ;
$$

the elements of $\mathbf{H}^1(t)$ are

$$
H_{ii}^1(t) = -\mu_i \frac{\gamma\, c_i}{\beta_i + c_i\, m_i(t)}\, \tilde{q}_i(\mathbf{m}(t))\, (1 - \tilde{q}_i(\mathbf{m}(t))) \quad ,
$$

and

$$
H_{ij}^1(t) = \mu_i \frac{\gamma\, c_j}{\beta_j + c_j\, m_j(t)}\, \tilde{q}_i(\mathbf{m}(t))\, \tilde{q}_j(\mathbf{m}(t)) \quad ,
$$

for $i \neq j$; $\mathbf{B}(t)$ is an $I \times I$ diagonal matrix with elements

$$
B_{ii}(t) = \sqrt{\lambda_i\, \alpha_v(t) + \mu_i\, \tilde{q}_i(\mathbf{m}(t)) \left(1 + 2\, \tilde{q}_i(\mathbf{m}(t)) \left\{ \mu_i \sum_j \frac{\tilde{q}_j(\mathbf{m}(t))}{\mu_j} - 1 \right\} \right)} \quad ;
$$

---

5 The Kronecker delta is defined by the relationship

$$
\delta(i, j) = \begin{cases} 1 & \text{if } i = j\,, \\ 0 & \text{if } i \neq j\,. \end{cases}
$$

$$\tilde{q}_i(\mathbf{m}(t)) = \frac{\frac{w_i}{\mu_i}\,[\beta_i + c_i\, m_i(t)]^{\gamma}}{\sum_j \frac{w_j}{\mu_j}\,[\beta_j + c_j\, m_j(t)]^{\gamma}} \quad ;$$

and $\mathbf{H}^2(t)$ has the column vector $(\lambda_1, \lambda_2, \dots, \lambda_I)'$ in the s^{th} column and zeros elsewhere;

$$s = \operatorname{argmin}\{\, (\alpha_1 - m_1(t)),\ \dots\ ,\ (\alpha_I - m_I(t)) \,\} \ .$$

As before, since (5.62) is of Ornstein-Uhlenbeck form, the variance-covariance matrix has elements which satisfy (5.27) and (5.28).

Example 5.8: As an example of an analysis using the aircraft detachment model with a repair policy that gives service priority based on which item has the lowest availability, i.e., PLA;p service, numerical examples with a common input and various solution methods are compared. Note that for the diffusion approximation, the determination of the controlling minimum availability is carried out within the numerical differential equation solver at every time step, so that no other distinction need be made between the cases considered in deriving the differential equations. The simulation uses the actual minimum as it moves along the sample path in each replication. The inputs for this example are shown in Figure 37.

i	1	2	3	4	5
K_i	100	110	120	130	140
λ_i	0.015	0.020	0.025	0.030	0.035
ν_i	1.0	1.1	1.2	1.3	1.4
w_i	1.0	1.0	1.0	1.0	1.0
$N(0)$	0	0	0	0	0

$A_C = 50$

Figure 37. Example 5.8 Inputs

Results for Example 5.8 are obtained and presented for each of the following cases.

a. Case APLA;1 (Diffusion Approximation, **PLA;1** Service). This case is the numerical solution obtained from the diffusion approximation in which the service rule is modeled by the probabilistic form

$$q_i(\mathbf{N}(t)) = \frac{w_i\,(K_i - N_i(t))^{-p}}{\sum_j w_j\,(K_j - N_j(t))^{-p}} \quad ; \tag{5.63}$$

with the parameter p set equal to 1.

b. Case APLA;p (Diffusion Approximation, **PLA;p** Service). This case is the same as Case APLA;1, but with the parameter p set equal to a high value, in this example 2, 4, 8, and finally 10, to get an analytical solution which approximates deterministic service of the item with the lowest availability.

c. Case SPLA;1 (Simulation, **PLA;1** Service). This case is the simulation outcome in which the service discipline is randomized selection of the next queue for service, upon each service completion, in accordance with probabilities using (5.63).

d. Case SLAF (Simulation, **LAF** Service). This case is the simulation outcome in which the service discipline is to select the item with the lowest availability to receive the next service upon each service completion.

Transient Results: For Example 5.8, the transient response of the system is summarized in tabular form in Figure 38, and then displayed graphically as a function of time. The tabulated results, showing the mean and standard deviation of the availability of each item as a function of time, are given at selected times for all the cases. The plots in Figure 39a. through e. show the solutions for Cases APLA;1 and SPLA;1, to compare diffusion approximation results with corresponding simulation results. The plots in Figure 40a. through e. show the solutions for Cases APLA;10 and SLAF, to compare diffusion approximation results with corresponding simulation of the LAF service discipline. Mean item availabilities and standard deviations were computed at unit time steps. Standard errors and confidence intervals for the point estimates for the means obtained from the simulation are omitted from the tabulated results to avoid more clutter in the table. Upper and lower .95 confidence limits for th[illegible]s are the point estimate $\pm$.0877 times the corresponding estimate for the standard deviation (i.e., about $\pm$ 10% of the standard deviation). Upper and lower .95 confidence limits for the standard deviations are .942 and 1.066 times the point estimate (i.e., about $\pm$ 5%).

Discussion of the Tabulated Results: At all t, the results show fairly good agreement between cases SPLA;1 and APLA;1, i.e., the diffusion approximation yields solutions close to the results from the simulation with probabilistic service. There is also fairly good agreement between cases APLA;10 and SLLF; i.e., the diffusion approximation

with a high power of p yields solutions close to the results from the simulation with service of lowest-availability-first. The agreement between means is better than the agreement between the standard deviations. This is seen better in the graphical comparisons. The lowest-availability-first discipline tends to drive the items toward equal availability in steady-state. This makes intuitive sense since whenever the availability for one particular item is less than the availability for the other items, it gets preferential service. At each time shown in the results, the effect of increasing the power p is seen to move the diffusion approximation results toward the LAF results.

	Means:					Standard Deviations:				
Item:	1	2	3	4	5	1	2	3	4	5
t = 50										
SPLA; 1	74.9	71.9	68.8	66.3	64.0	6.6	7.7	8.4	8.6	9.5
APLA, 1	74.9	72.1	69.2	66.5	63.7	6.6	7.4	8.1	8.8	9.4
APLA; 2	75.4	72.2	69.2	66.2	63.3	6.3	7.0	7.7	8.3	8.8
APLA; 4	76.0	72.4	69.0	65.8	62.8	5.9	6.5	7.0	7.5	7.9
APLA; 8	76.7	72.4	68.6	65.4	62.6	5.4	5.8	6.1	6.4	6.6
APLA; 10	77.1	72.0	68.0	64.9	62.3	5.4	5.6	5.8	6.0	6.2
SLAF	75.7	70.8	67.7	65.9	64.7	4.7	5.2	4.9	4.8	5.0
t = 100										
SPLA; 1	58.6	48.3	39.0	30.2	22.2	8.4	9.4	10.3	9.0	6.4
APLA; 1	57.8	47.4	37.5	28.2	19.9	7.9	8.7	9.1	9.2	4.0
APLA; 2	55.2	44.9	35.5	27.7	21.9	7.2	7.7	7.6	7.0	3.7
APLA; 4	51.9	41.2	32.7	27.4	24.6	6.7	6.7	5.9	4.8	3.7
APLA; 8	49.3	37.4	30.3	27.8	26.7	6.7	6.1	4.6	4.1	3.8
APLA; 10	48.8	35.9	29.4	27.7	26.8	6.8	6.0	4.3	4.0	3.8
SLAF	46.4	34.3	29.2	28.0	27.2	6.9	5.3	3.6	3.3	3.1
t = 150										
SPLA; 1	53.3	40.8	30.5	21.7	15.6	8.9	9.4	9.1	7.3	4.1
APLA; 1	52.1	39.5	28.3	19.4	13.8	8.1	8.4	8.1	6.8	2.9
APLA; 2	45.0	32.7	23.5	18.4	16.0	7.3	7.0	5.7	4.4	3.1
APLA; 4	37.9	25.2	19.5	17.7	16.8	7.0	5.5	3.7	3.3	3.0
APLA; 8	33.8	20.2	18.1	17.5	17.1	7.4	4.0	3.2	3.1	3.0
APLA; 10	33.3	18.9	17.7	17.2	16.8	7.6	3.5	3.1	3.0	3.0
SLAF	31.1	19.5	18.0	17.5	17.0	8.1	3.3	2.4	2.3	2.2

Figure 38a. Example 5.8 Item Availability; Transient

	Means:					Standard Deviations:				
Item:	1	2	3	4	5	1	2	3	4	5
t = 200										
SPLA; 1	49.3	35.7	25.6	18.2	14.1	9.1	8.8	8.0	5.8	3.5
APLA; 1	48.0	34.1	23.0	16.0	12.5	8.1	8.0	6.9	5.1	2.8
APLA; 2	37.0	24.4	17.6	15.0	13.6	7.1	5.8	4.2	3.5	2.9
APLA; 4	27.8	17.2	15.1	14.3	13.6	6.8	3.6	3.0	2.9	2.7
APLA; 8	22.9	14.6	14.0	13.6	13.3	7.6	2.8	2.7	2.7	2.6
APLA; 10	22.4	14.0	13.6	13.3	13.1	8.1	2.8	2.8	2.7	2.6
SLAF	21.6	14.8	14.3	13.9	13.5	6.9	1.9	1.9	2.0	1.9
t = 250										
SPLA; 1	45.9	31.8	22.7	16.4	12.9	8.7	8.2	7.2	4.8	3.2
APLA; 1	44.3	29.8	19.7	14.4	11.9	8.1	7.5	5.9	4.4	2.8
APLA; 2	30.6	19.3	15.1	13.4	12.3	6.7	4.7	3.5	3.3	2.6
APLA; 4	20.2	14.2	13.2	12.5	12.0	5.8	2.9	2.7	2.6	2.5
APLA; 8	14.8	12.7	12.3	12.0	11.8	5.1	2.5	2.5	2.4	2.4
APLA; 10	14.1	12.5	12.2	11.9	11.7	5.3	2.6	2.5	2.5	2.4
SLAF	15.8	13.2	12.8	12.4	12.1	4.7	1.7	1.8	1.7	1.8
t = 300										
SPLA; 1	43.3	29.1	20.3	15.4	12.7	8.6	7.4	6.1	4.4	3.1
APLA; 1	41.0	26.5	17.6	13.5	11.4	8.0	6.9	5.1	4.2	2.8
APLA; 2	25.3	16.4	13.8	12.4	11.4	6.1	3.8	3.3	3.1	2.5
APLA; 4	15.3	12.9	12.1	11.5	11.0	3.8	2.6	2.6	2.5	2.4
APLA; 8	12.0	11.4	11.1	10.8	10.6	2.6	2.4	2.4	2.3	2.3
APLA; 10	11.6	11.2	10.9	10.7	10.6	2.6	2.5	2.4	2.4	2.4
SLAF	13.4	12.3	11.9	11.6	11.4	2.6	1.7	1.7	1.7	1.7
t = 350										
SPLA; 1	41.0	26.7	19.3	14.9	12.4	8.3	7.3	5.5	4.1	3.3
APLA; 1	38.1	23.9	16.4	13.0	11.0	7.8	6.3	4.7	4.0	2.7
APLA; 2	21.4	15.0	13.0	11.8	10.9	5.2	3.4	3.1	3.0	2.5
APLA; 4	13.3	12.0	11.3	10.8	10.4	2.8	2.5	2.5	2.4	2.2
APLA; 8	11.3	10.8	10.5	10.3	10.1	2.4	2.3	2.3	2.2	2.2
APLA; 10	11.0	10.7	10.4	10.2	10.1	2.5	2.4	2.3	2.3	2.3
SLAF	12.4	11.9	11.5	11.2	11.0	1.7	1.5	1.6	1.6	1.7
t = 400										
SPLA; 1	38.7	25.2	18.6	14.7	12.5	7.9	6.3	5.2	3.9	3.4
APLA; 1	35.5	22.0	15.6	12.6	10.8	7.6	5.8	4.5	4.0	2.6
APLA; 2	18.7	14.2	12.5	11.4	10.5	4.4	3.3	3.0	2.9	2.4
APLA; 4	12.6	11.6	11.0	10.5	10.0	2.6	2.5	2.4	2.3	2.2
APLA; 8	11.0	10.6	10.3	10.1	9.9	2.4	2.3	2.2	2.2	2.2
APLA; 10	10.8	10.5	10.2	10.0	9.9	2.4	2.4	2.3	2.3	2.2
SLAF	12.2	11.7	11.4	11.2	10.8	1.5	1.5	1.6	1.5	1.6

Figure 38b. Example 5.8 Item Availability; Transient (cont.)

Discussion of the Graphical Comparisons: As seen in all the plots the agreement between means is better than the agreement between the standard deviations. A clear feature in the plots of standard deviations in Figure 39a. through e., which compare Cases APLA;1 and SPLA;1, is the point at which the diffusion approximation and the simulation results separate. That point occurs at the same time, $t = 60$, for all items. That time corresponds to the point at which the availability of item 5 dropped below $A_c = 50$, and the system of differential equations governing the system changed. Although the absolute and relative errors between the diffusion approximation and simulation are much greater than in the previous model, especially in the standard deviations, the diffusion approximation curves do roughly follow the shape of the simulation response providing a usable approximation, even in the worst cases seen in Figure 39e. and Figure 40a. The agreement in the means is much better than the standard deviations. Some separation in means is seen in Figure 39a. through e., although this, too, shows a usable approximation.

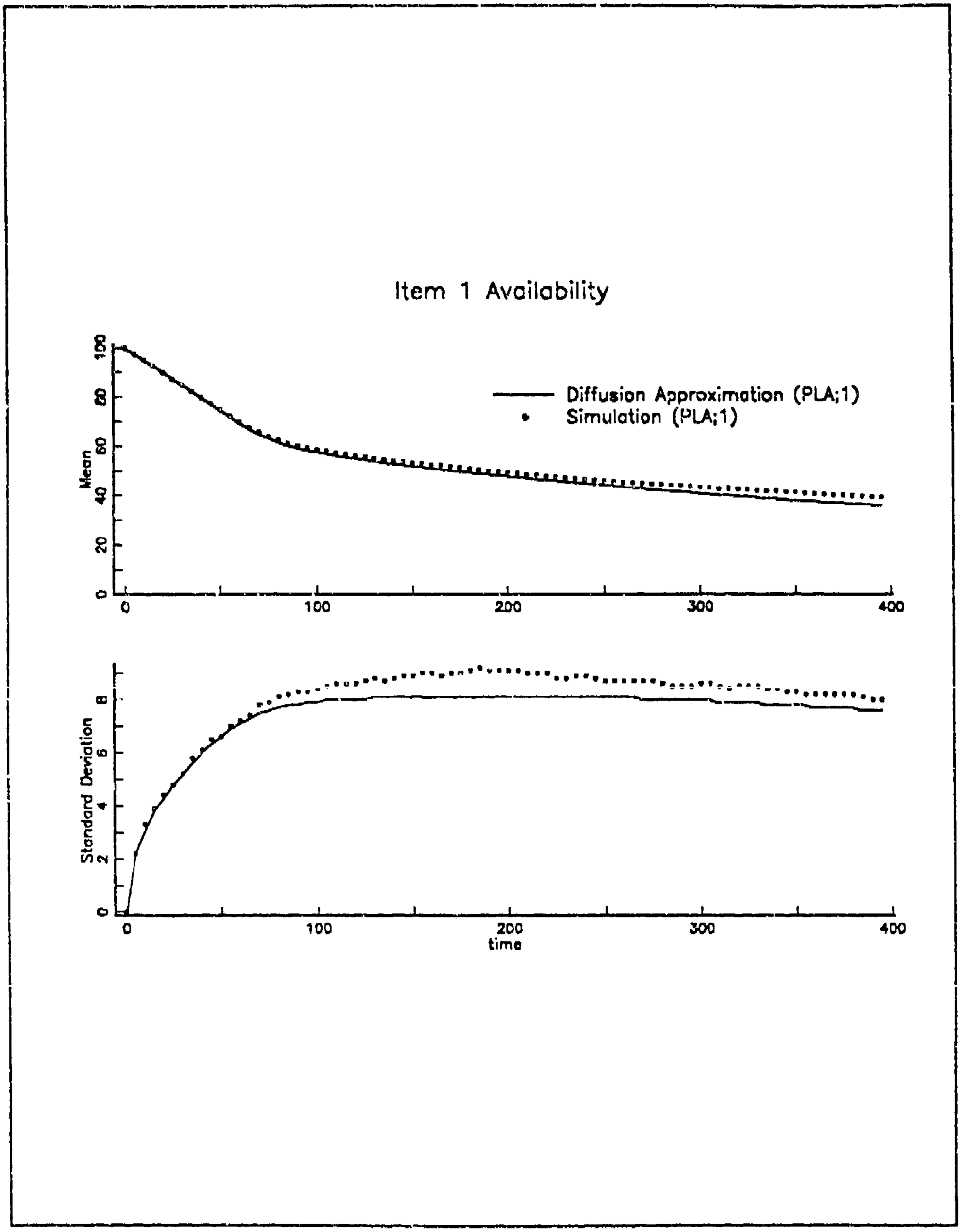


Figure 39a. Example 5.8 Item 1 Availability, PLA;1 (transient)

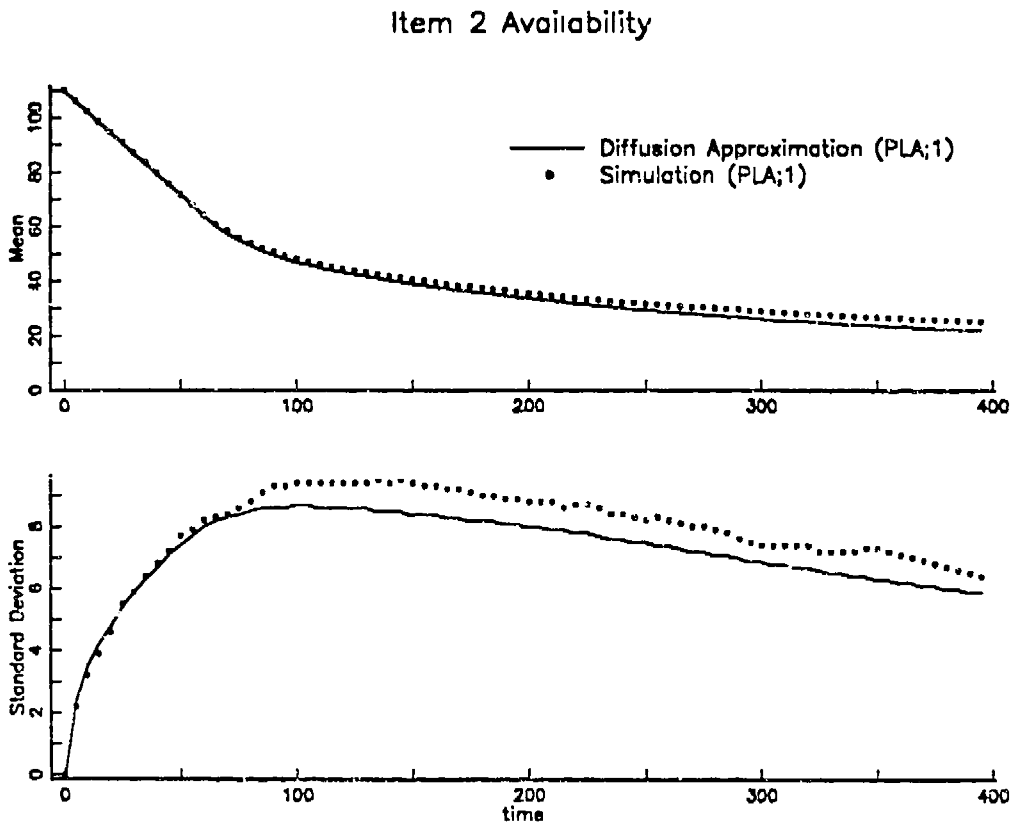


Figure 39b. Example 5.8 Item 2 Availability, PLA;1 (transient)

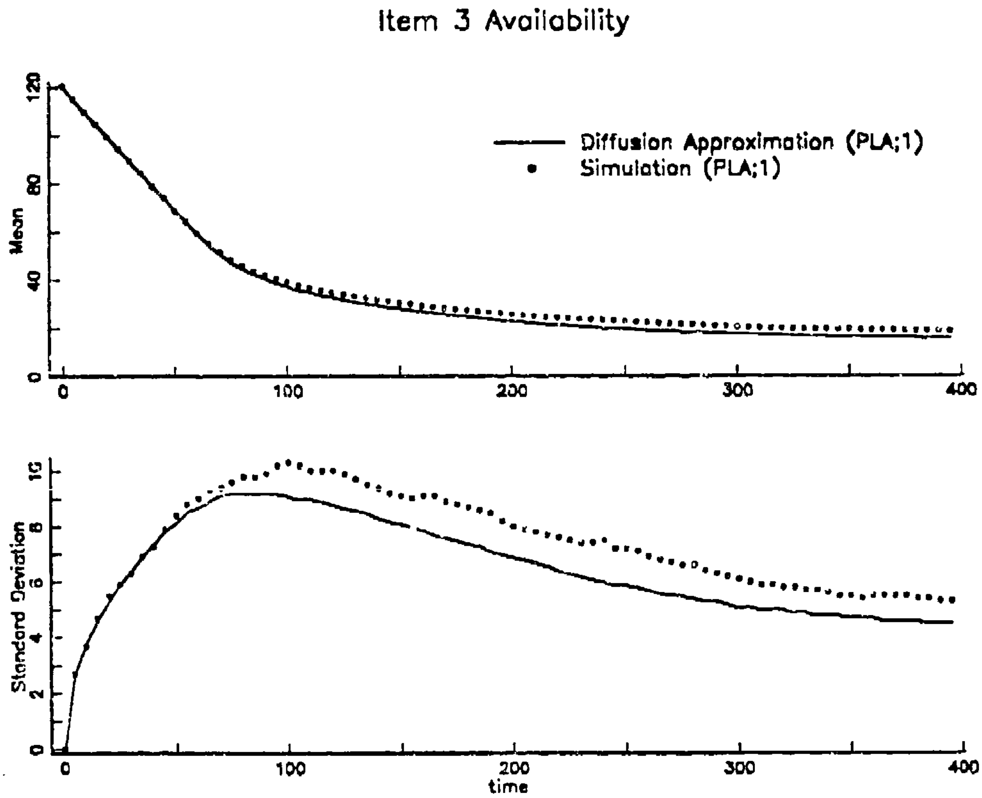


Figure 39c. Example 5.8 Item 3 Availal ility, PLA;1 (transient)

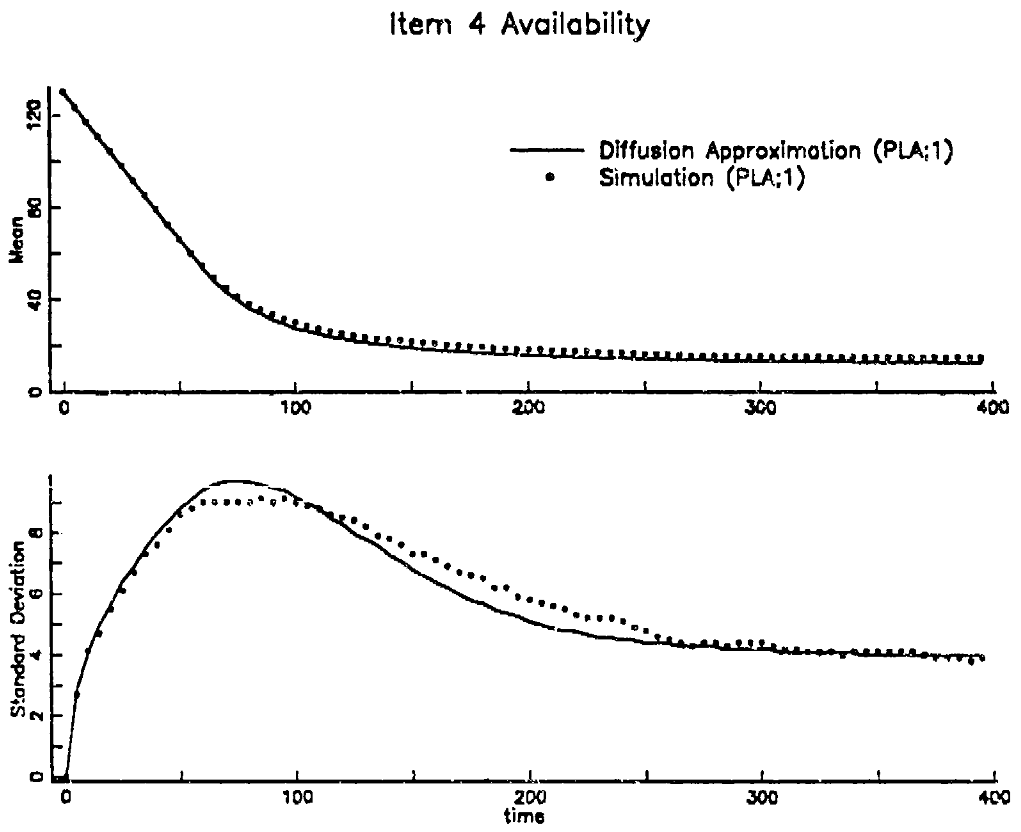


Figure 39d. Example 5.8 Item 4 Availability, PLA;1 (transient)

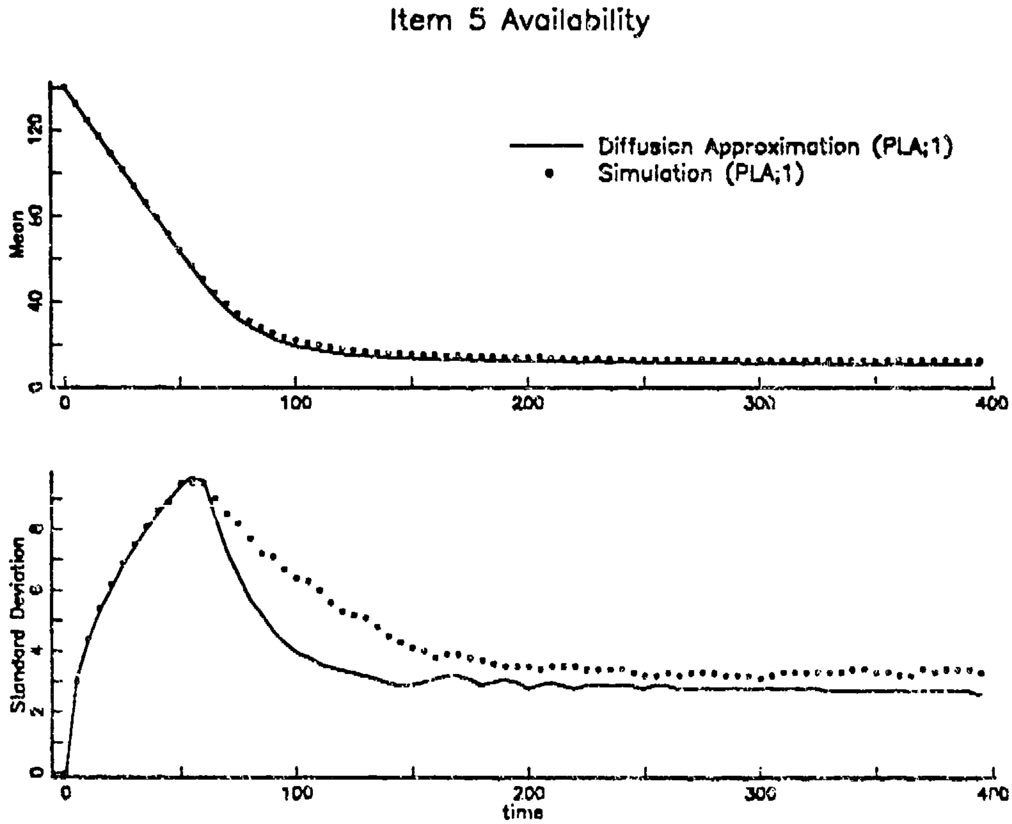


Figure 39e. Example 5.8 Item 5 Availability, PLA;1 (transient)

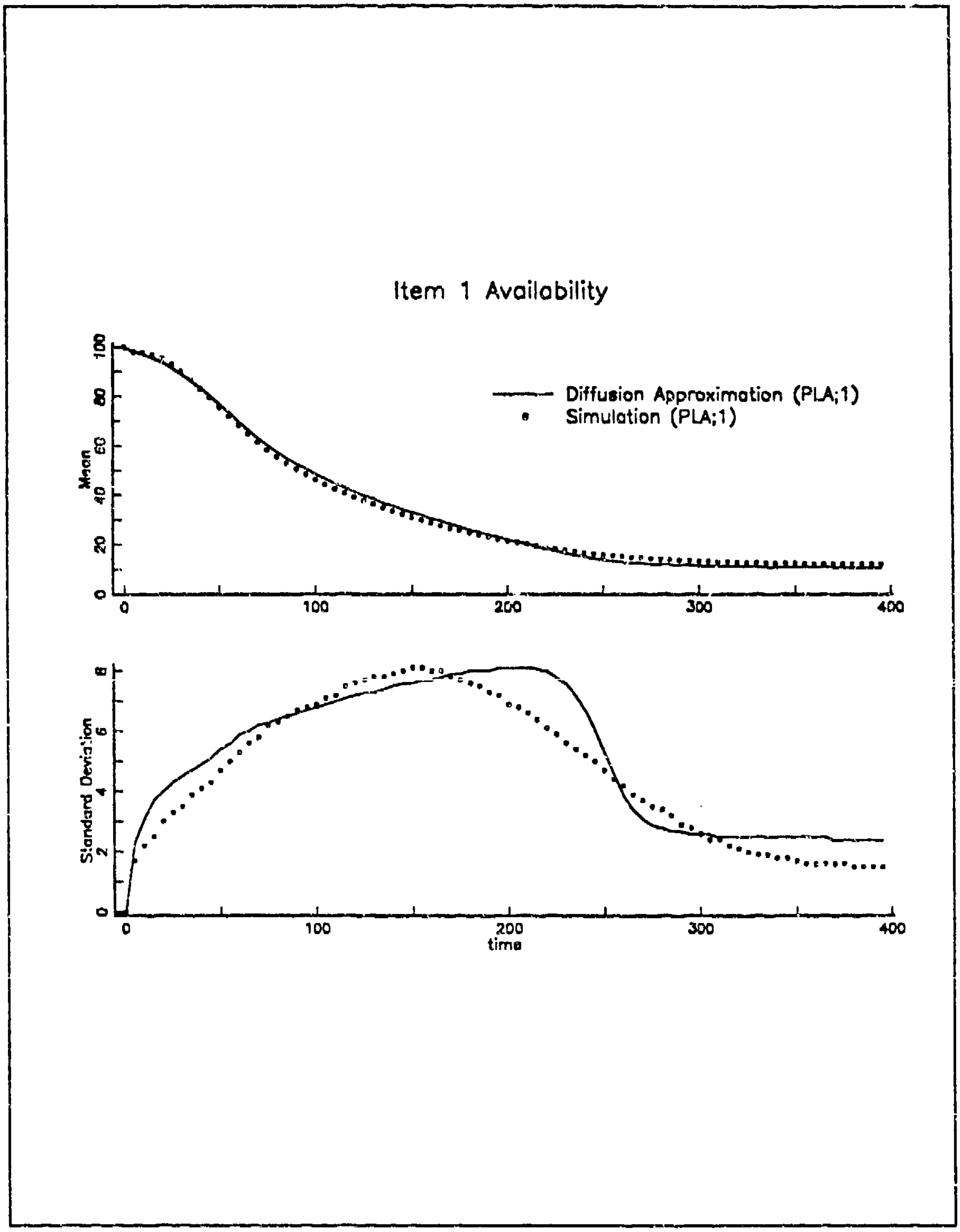


Figure 40a. Example 5.8 Item 1 Availability, PLA;10 (transient)

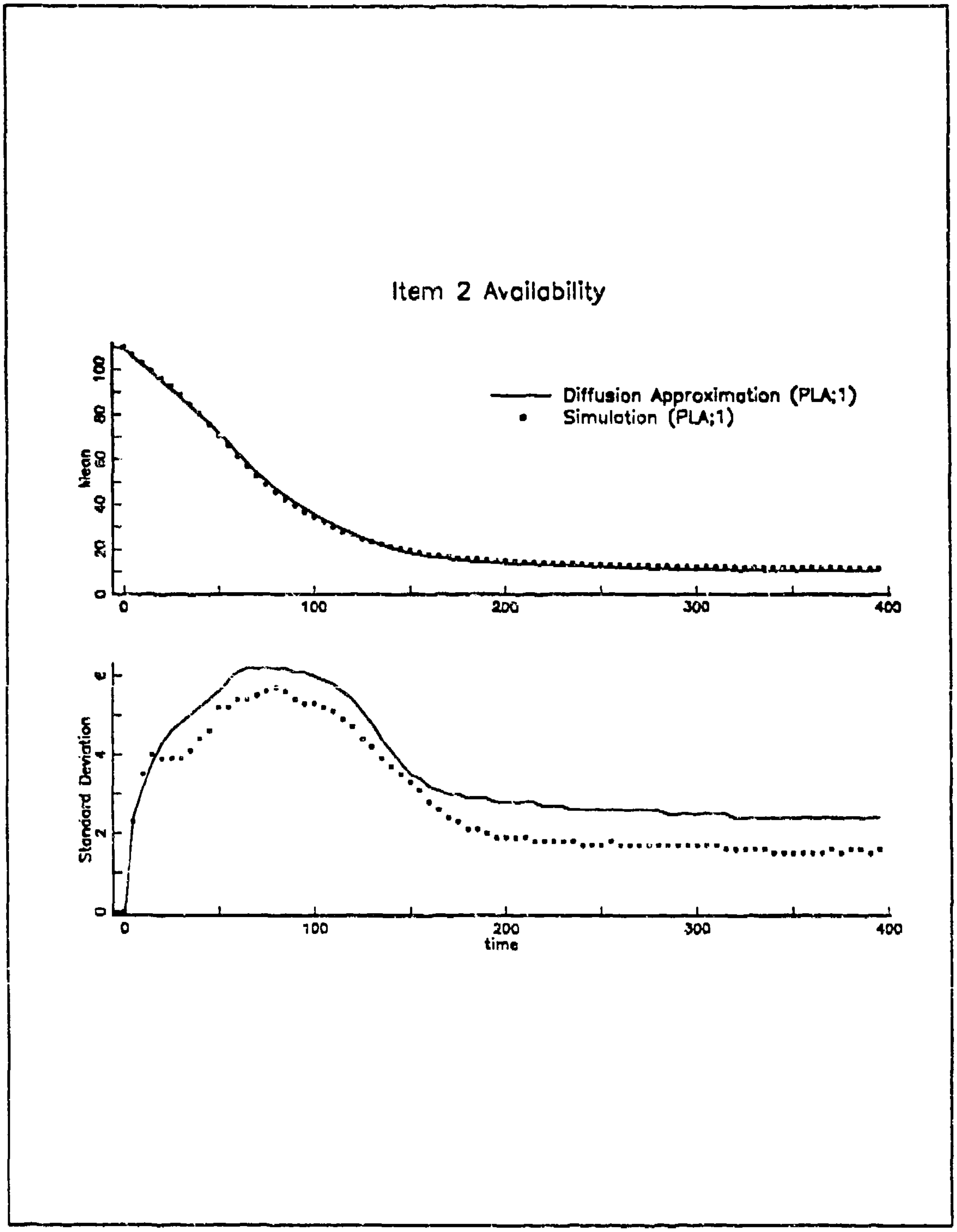


Figure 40b. Example 5.8 Item 2 Availability, PLA;10 (transient)

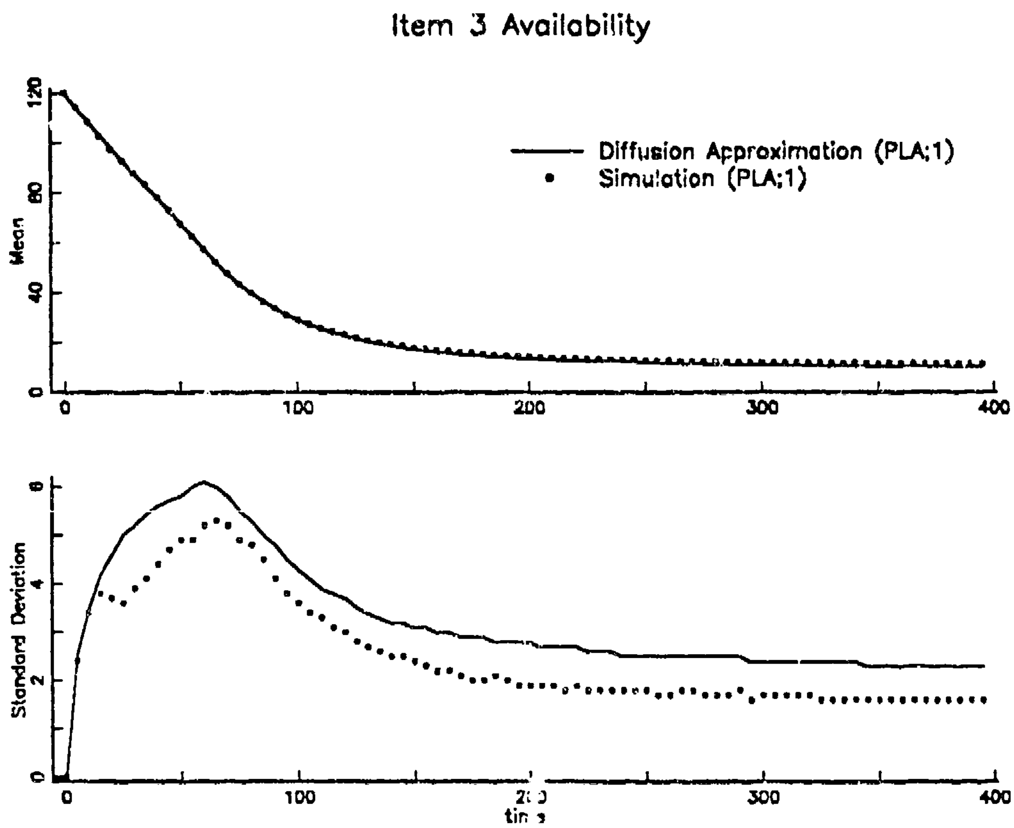


Figure 40c. Example 5.8 Item 3 Availability, PLA;10 (transient)

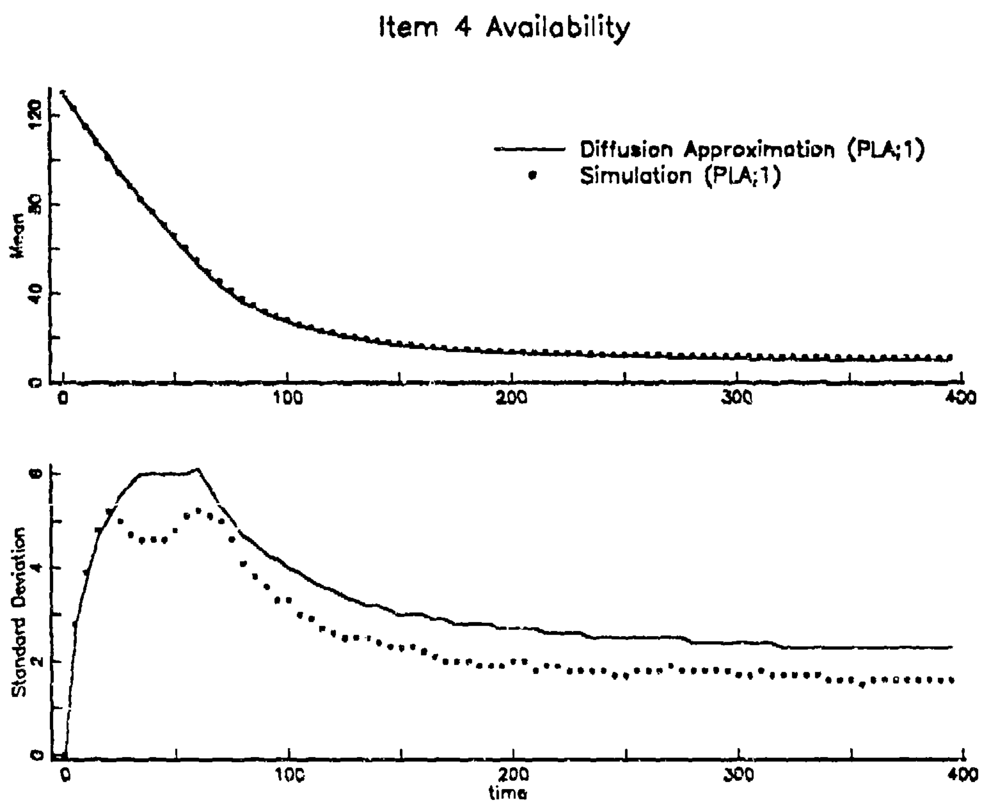


Figure 40d. Example 5.8 Item 4 Availability, PLA;10 (transient)

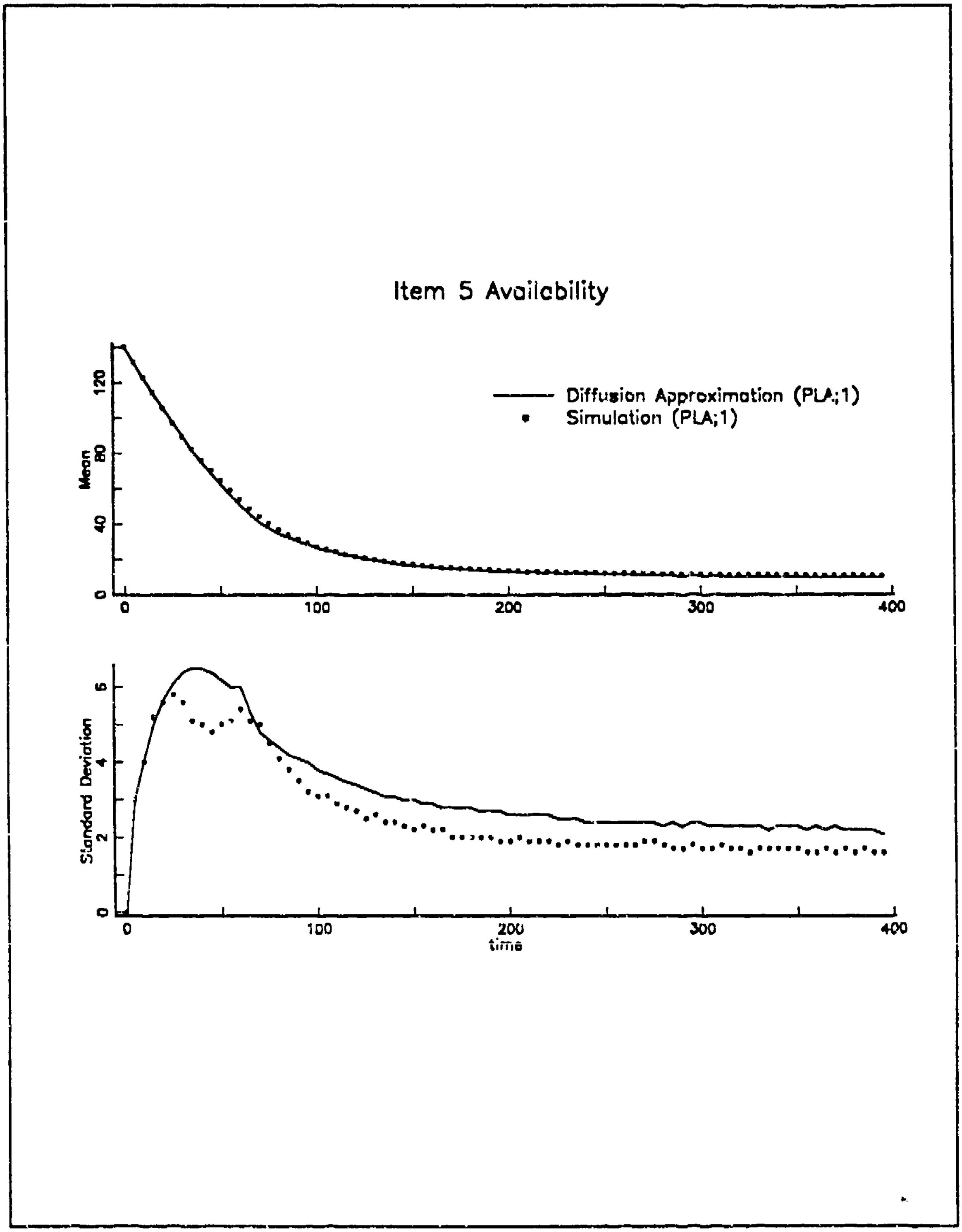


Figure 40e. Example 5.8 Item 5 Availability, PLA;10 (transient)

Steady-state Results: For Example 5.8, steady-state numerical results are summarized in Figure 41a. and b.. As discussed previously, the lowest-availability-first discipline tends to drive the items toward equal availability in steady-state. An interesting phenomenon which distinguishes the aircraft detachment model from the previous repairman model is evident in these results. In this model, the common value approached under LAF is the same minimum availability as in PLA;1, rather than some average value as happened in the previous repairman model.

Means:

Item	APLA; 1	SPLA; 1	.95 C. I.
1	22.91	27.62	27.44 , 27.79
2	17.15	20.53	20.36 , 20.69
3	13.72	16.49	16.40 , 16.57
4	11.43	13.74	13.60 , 13.87
5	9.80	11.57	11.47 , 11.68

Standard Deviations:

Item	APLA; 1	SPLA; 1	.95 C. I.
1	5.12	5.61	5.50 , 5.73
2	4.48	4.87	4.80 , 4.95
3	4.04	4.33	4.28 , 4.38
4	3.72	3.76	3.69 , 3.83
5	2.51	3.08	3.04 , 3.11

Figure 41a. Example 5.8 Steady-state Summary: PLA;1

Means:

Item	APLA; 10	SLAF	.95 C. I.
1	10.66	12.06	11.96 , 12.17
2	10.36	11.69	11.60 , 11.77
3	10.13	11.35	11.35 , 11.35
4	9.95	11.03	10.97 , 11.08
5	9.80	10.71	10.61 , 10.81

Standard Deviations:

Item	APLA; 10	SLAF	.95 C. I.
1	2.42	1.51	1.49 , 1.52
2	2.35	1.54	1.52 , 1.55
3	2.30	1.57	1.55 , 1.58
4	2.26	1.60	1.58 , 1.61
5	2.23	1.62	1.61 , 1.63

Figure 41b. Example 5.8 Steady-state Summary: PLA;10, LAF

Since the solution of the system of differential equations is computationally fast, and could be conveniently done on a small computer, and provides usable approximations of the transient behavior of the system, the diffusion approximation may be used to examine various service disciplines, and try various heuristic weights, w_i, to find the best policy with respect to a measure of effectiveness that uses the mean and variance of the number of items awaiting repair at any time, or equivalently, the aircraft availability at any time.

Areas for further study in combat logistics support are discussed in the conclusions.

VI. CONCLUSIONS AND FURTHER STUDY

A. OPERATIONAL COMBAT LOGISTICS

In the part of this thesis concerning operational combat logistics, a problem of operational interest in the Navy was defined and studied. The problem was to schedule the replenishment of weapons within a Navy Battle Group following a combat engagement, when the uncertain arrival of another attack may interrupt the replenishment process before all requirements are satisfied.

The concept of combat logistics objectives as a fusion of pure combat and pure logistics objectives was introduced. The idea of dealing with an operational combat logistics problem also came into consideration when choosing units of measurement.

Simple stochastic optimization models were developed for the combat VERTREP problem and the combat CONREP problem. Methodologies were developed for calculating the time it takes to conduct ammunition replenishments, and for quantifying the combat value of weapons in a Battle Group in a way that is useful for scheduling replenishment.

In one simple model, optimal Vertical Replenishment scheduling was achieved with a dynamic allocation index, called Logistics Weighted Combat Value (LWCV). The LWCV method was then used in an efficient scheduling heuristic for a realistic model and produced results which compared very favorably with a locally optimum schedule obtained with a lengthy local neighborhood search.

In a separate simple model, optimal Connected Replenishment scheduling was achieved with dynamic programming (DP). The DP approach was then adapted to more realistic situations.

1. Further Research

a. Heuristic Improvement of LWCV VERTREP Schedule

It was noted that the use of exhaustive all-pairs and all-triples interchange improvement searches were certainly not t he only alternatives to staying with the initial schedule obtained with the LWCV heuristic. Although many general improvement techniques from the extensive combinatorial optimization literature could be tried, con-

sideration of the special characteristics of the VERTREP problem suggests investigating heuristic improvement methods tailored to the problem.

It appears that the aspect of the general Battle Group VERTREP problem that has the greatest impact with respect to the optimality of the LWCV procedure is the occurrence of strikedown queues. The Battle Group VERTREP example of Chapter III demonstrates that a very significant qualitative difference between the initial and k-opt schedules can be attributed to when lifts of WepA are dispatched to Ship1. In that example, that weapon had both the highest combat potential and longest strikedown time in the Battle Group. Under the LWCV procedure, the long strikedown times led to the formation of strikedown queues which ultimately delayed the accrual of combat value of lifts backlogged on the deck of the receiver. Since the LWCV heuristic uses a forward induction policy, it can not look ahead to avoid that complication. The k-opt schedules, on the other hand, through a succession of interchanges, spread out the delivery of that weapon, trading off some early helicopter delivery delay to obtain a sequence which is more efficient by avoiding the *wasted time* that weapons would have spent in strikedown queues.

A possible improvement heuristic that should be investigated is based on the foregoing observation that lifts with high combat value and long strikedown times need special consideration in scheduling. One heuristic would be to search for improvements due to *insertion* of such a lift earlier in the sequence than the LWCV heuristic provides. Many variants of this idea are easily conceived.

An important consideration in any improvement strategy for the Combat VERTREP problem should be how long it takes to find notable improvements. It is conceivable that a user might specify an upper time limit for which he is willing to wait for improvements to an initial schedule. For example, if a good initial schedule is obtained in three seconds, the user may be willing to allow up to ten times that, or thirty seconds, for heuristic improvement. In such a situation, it would be particularly desirable to try to find a few high return improvements early.

b. Dynamic Revision of CONREP with Strikedown Queues

In Chapter IV, the idea of dynamic revision of a CONREP schedule was developed through the case of interrupting a receiver in progress and rescheduling under the assumption that strikedown queues did not occur for the receiver being interrupted. An important extension of this model is to consider the situation when strikedown

queues do occur. One possible approach to this problem that is left for further investigation is outlined here.

(1) The reason for the simplifying assumption that strikedown queues could not occur was so that any positive delay between when a receiver in progress was interrupted and when he was scheduled to start again would not affect strikedown completion times. In contrast to this, if strikedown queues did develop, the receiver processing times would include waiting time in the strikedown queue. Then a positive interruption delay would allow the strikedown queue to shorten (or empty), and hence reduce the receiver processing times following a delay. Furthermore, the amount by which the subsequent strikedown times are shortened would depend on the length of the delay, which would not be known until after rescheduling is finished.

(2) An insight, which suggests a way to proceed, is to observe that although the length of the delay is variable, and consequently the amount by which the subsequent strikedown times are shortened is variable, at the point when an interruption occurs, the event time when a strikedown queue will empty is fixed. To be specific, say there is only one weapon system with a strikedown queue, and let δ^* denote the length of time from when an interruption occurs until that strikedown queue empties. Also, let δ denote the delay from when an interruption occurs until the rescheduled receiver is delivered another lift of that weapon.

(3) The presence of a strikedown queue may be thought of as an additional constraint on the interruption with rescheduling problem, and the method introduced for the special case of no strikedown queues can be thought of as a relaxation of this additional constraint. A first step for the problem with strikedown queues is to solve the relaxation using Proposition 5.5 under the following conditions:

(a) Pseudo-receiver ja_0 strikedown completion times for subsequent lifts are not modified (i.e., include waiting time due to existing strikedown queue), since the scheduling of ja_0 represents the receiver in progress continuing without interruption. This is the actual condition without any relaxation.

(b) Pseudo-receiver ja_1 strikedown completion times assume no queue exists as of when ja_1 starts service. This is a relaxation, because this is the condition for $\delta \geq \delta^*$, but is optimistic for $\delta < \delta^*$.

(4) The solution of the relaxation will fall into one of the three following cases:

(a) The receiver in progress is scheduled to continue without delay (i.e., ja_0 is scheduled to commence immediately). In this case, the occurrence of a strikedown queue has been accounted for, since the strikedown times were not modified.

(b) The receiver in progress is actually interrupted and rescheduled (i.e., ja_1 is scheduled) following some delay such that $\delta \geq \delta^*$. In this case, the previous occurrence of a strikedown queue has been accounted for, since the delay has allowed the queue to empty.

(c) Pseudo-receiver ja_1 is scheduled, however the delay is such that $\delta < \delta^*$. In this case, the previous occurrence of a strikedown queue has not been properly accounted for, since the strikedown completion times were overly optimistic.

(5) For cases (a) and (b) above, the relaxation provided a schedule which is feasible for the original problem and is thus optimal. However, for case (c), more work is required.

(6) A possible approach is to consider modifications to the backward induction of Proposition 4.5. The first modification must be that for states which include pseudo-receiver ja_1, the expressions for expected combat value contributions should be modified to reflect the fact that value for subsequent lifts can not accrue until after the fixed time when the previous strikedown queue empties. The second modification concerns how additional receivers are added to a state which includes ja_1 -- the key point being how additional receivers and lifts are *packed* in before ja_1. The packings fall into two categories. The first, which is related to case (b) above, concerns those combinations of receivers and lifts such that $\delta \geq \delta^*$, in which case ja_1 strikedown completion times are referenced to δ. The second category concerns those combinations of receivers and lifts such that $\delta < \delta^*$, in which case ja_1 strikedown completion times are referenced to δ^*.

Working out the details of such an approach is an area for further study.

c. Combined CONREP and VERTREP Scheduling

The issue of finding a combined schedule for simultaneous CONREP and VERTREP requires further study. The separate models for scheduling VERTREP and CONREP may be combined heuristically as an initial approach. However, other than some intuitive appeal as a means of getting *a* schedule, no firm justification is offered. Additional investigation is needed.

One possibility for heuristically combining the separate VERTREP and CONREP scheduling procedures is to consider a stepwise application of the two meth-

ods. An example of a stepwise method is outlined in the following, for which it is assumed that maneuvering orders have been issued to direct the receivers to close the delivery ship at best speed to a position where VERTREP flight time is minimized, and where the receiver is ready for CONREP if so ordered.

(1) As the receivers are closing the delivery ship, an initial VERTREP schedule is generated and implemented through the time that the first receiver(s) are in a position where they could commence CONREP.

(2) As of that time, the ammunition requests from each receiver are decremented by projected VERTREP deliveries, corresponding ammunition strikedown times are updated, and an initial CONREP schedule is generated for those receivers in position.

(3) With the scheduled CONREP deliveries being accounted for, the VERTREP scheduling is continued, until another receiver is available to commence CONREP.

(4) As of that time, dynamic CONREP schedule revision is computed.

(5) Step (3) and (4) are repeated until the entire combined schedule is generated.

d. Combat Value Functions

Although the use of a utility scale for quantifying combat value is justified by the complexity of the combat scenarios facing a Battle Group, the Priority List Method presented in Appendix B is merely one possible approach. Further research is needed concerning the quantification of combat value functions for use in operational combat logistics models.

e. Implementation in a Decision Support System

The actual implementation of the models for scheduling ammunition replenishment during combat, including an appropriate user interface, is an area for further work. It is envisioned that such an implementation would be in the form of scheduling modules embedded in a larger Decision Support System available to the Battle Group Commander.

The development of a Decision Support System to support operational combat logistics at the Battle Group and Battle Force levels is the subject of ongoing work at the Naval Postgraduate School by Schrady and Wadsworth.

f. Optimal Maneuvering

The issue of optimal maneuvering tactics for replenishment requires further study. At best, the methodology in this work allows maneuvering variables to be treated parametrically, so that various options could be considered using a *what if* approach. It is clear that CONREP and VERTREP schedules depend on the maneuvering tactics; and it is also clear that a good maneuvering plan depends on the combat value achievable in the replenishment. However, determination of the best maneuvering plan in the combat replenishment situation is really a *multiple objective optimization problem.* Besides the objective of maximizing the combat value of ammunition transferred before raid arrival, a very important objective may be tactically motivated. For example: minimizing the exposure of the delivery ship and other high value units to submarine torpedo attack; or, if the Battle Group is withdrawing from the area due to damage from the last raid, maximizing the Battle Group movement in the direction of withdrawal. In addition, the maneuvering problem has constraints which include wind, sea state, and remaining in navigable waters. The problem of how to maneuver replenishment and combatant ships is the subject of ongoing work at the Naval Postgraduate School by Hardgrave and Lawphongpanich.

g. Ordnance on Deck

Although the models considered in this work do not preclude decreasing combat value functions (i.e., negative marginal utility), they do not capture the possible loss in combat effectiveness due to logistics if a raid arrives while a strikedown is in progress, catching the receiver with weapons on deck. Besides not being ready for combat, ordnance on deck is vulnerable during an attack and may constitute a secondary explosion hazard. Put another way, the models allow combat value to decrease deterministically due to assumptions of a pure combat model, but do not allow combat value to decrease stochastically due to logistics.

For future study, a possible way to capture this *real world* consideration, is to incorporate a combat value penalty function which probabilistically decreases combat value if a raid arrives and finds weapons on deck. For example, using the notation of Chapter IV, expressions for expected total combat value, now include terms such as:

$$\dots + v_i P[T \geq c_i] \dots$$

which represent the expected accrual of marginal combat value v_i if a raid arrival, T, is later than the time of strikedown completion, c_i. Additional *penalty* terms which might also be included could take the form:

$$\ldots \quad - z_i \, \mathrm{P}[x_i \leq T \leq c_i] \quad \ldots$$

for penalties $z_i \geq 0$. Such terms could represent an explicit penalty if the raid arrival came after the time of delivery, x_i, but before the time of strikedown completion. The idea of penalizing strikedown queues requires further investigation.

B. COMBAT SUPPORT LOGISTICS

In the part of this thesis concerning combat support logistics, a methodology was developed for analyzing the transient behavior of a service system for a large population of modules under heavy traffic conditions where service policies with queue-length influence are used. The modeling technique used a diffusion approximation valid for the heavy traffic conditions anticipated under combat conditions. The analytic solutions, which were obtained very quickly, were compared to simulation results and found to be very satisfactory.

Alternative scheduling policies that reflect different organizational maintenance service disciplines can be readily compared. The model also provides a framework for choosing near-optimal spare module allocation within budget constraints.

Besides starting the transient analysis from a known state (with zero variance), the methodology is also applicable to initial conditions with arbitrary queue length mean and variance. For example, the model may be used to analyze the response of the repair shop during a future transition from peacetime to combat conditions. In this case, the steady-state attained under peacetime conditions provides queue length mean and variance initial conditions for the transient analysis under combat conditions.

1. Further Research

The combat support logistics models considered various repair shop disciplines and general service time distributions for a single server. An area for future research concerns extensions to multiple servers.

The failure processes considered were for parts with individual Markovian failures. The failure rates seen at the repair facility were proportional to either the total number of parts in the population less the number awaiting or undergoing repair, for the

repairman model, or the number of operational aircraft, for the aircraft detachment repairman model. In both situations, the probability of multiple failures in the interval $(t, t+dt)$ is $o(dt)$.

Another area for future research concerns modeling the possibility of catastrophic failures which would cause group arrivals at the repair shop. Two situations are envisioned for group arrivals due to the Markovian occurrence of a catastrophic event. In the first situation, a catastrophic event causes either zero or one item failure of each type simultaneously (i.e., given a catastrophic event in the interval $(t, t+dt)$, which occurs with probability $\lambda_C dt + o(dt)$, the number of items of type i which fail due to that catastrophic event is a Bernoulli random variable with parameter p_i). The current diffusion approximation model may be readily adapted to this situation. The second situation involves multiple failures of each type (i.e., given a catastrophic event at time t, the number of items of type i which fail due to that catastrophic event is a binomial random variable with parameters p_i and $n_i = [K_i - N_i(t)]$). Preliminary work with this situation indicates that, for large systems in heavy traffic, accurate results may be obtained for the mean queue lengths using ordinary differential equations, but that the random fluctuations about the deterministic mean do not converge to a diffusion. Further research is needed to model catastrophic event failures.

APPENDIX A. PROTOTYPE VERTREP SEQUENCING PROGRAM

```
* * * * * Variable definitions * * * * *
* * Index usage:
*     L          Lifts
*     R          Receivers
*     RSTAR      Indes of optimal rcvr to get next lift
* * Given data
*     TAU        Expected value of time available
*     RCVRS      Number of receivers
*     N(R)       Number of lifts requested by rcvr R
*     SO(R)      Initial weapons state of rcvr R
*     DELTA(R)   Time for helo to deliver lift to rcvr R
*     RHO(R)     Time for helo to return from rcvr R
*     SIGMA(R)   Time for rcvr R to strikedown a lift
*     P(R)       Single shot Pk of missiles on rcvr R
*     PI(R)      Prob. attacker is engageable by rcvr R
* * Other variables
*     LSUM       Total number of lifts requested
*     S(R)       Weapons state of rcvr R
*     SMAX(R)    Maximum weapons state of rcvr R
*     C(R)       Constant derived from given data
*     PBAR(R)    Derived constant (1-P(R))
*     LWCV(R)    Logistics Weighted Combat Value
*     BEST       Maximum LWCV(R)
* * * * * Variable declarations * * * * *
      REAL TAU,P(9),PI(9),DELTA(9),RHO(9),SIGMA(9)
      REAL C(9),PBAR(9),BEST,LWCV(9)
      INTEGER R,L,RCVRS,LSUM,N(9),S(9),SO(9),SMAX(9),RSTAR
* * * * * Initialize and read data * * * * *
      READ(5,*)TAU,RCVRS
      LSUM=0
      DO 10 R=1,RCVRS
         READ(5,*)N(R),SO(R),DELTA(R),RHO(R),SIGMA(R),P(R),PI(R)
         LSUM=LSUM+N(R)
         S(R)=SO(R)
         SMAX(R)=SO(R)+N(R)
         C(R)=PI(R)*P(R)*EXP(-(DELTA(R)+SIGMA(R))
     +         /TAU)/(1.-EXP(-(DELTA(R)+RHO(R))/TAU))
         PBAR(R)=1.-P(R)
         LWCV(R)=C(R)*PBAR(R)**S(R)
   10 CONTINUE
      L=1
* * * * * Main loop * * * * *
      DO 20 L=1,LSUM
         BEST=0.
         DO 30 R=1,RCVRS
            IF(S(R).LT.SMAX(R))THEN
               IF(LWCV(R).GT.BEST)THEN
                  RSTAR=R
                  BEST=LWCV(R)
```

```
            ENDIF
          ENDIF
 30     CONTINUE
        WRITE(6,100)RSTAR
        S(RSTAR)=S(RSTAR)+1
        LWCV(RSTAR)=C(RSTAR)*PBAR(RSTAR)**S(RSTAR)
 20   CONTINUE
      STOP
100     FORMAT(I10)
      END
```

APPENDIX B. COMBAT VALUE

In this appendix, the concept of *combat value* as a measure of the utility of ammunition to a Battle Group facing combat is discussed. The very simple combat model introduced in the prototype model of Chapter III is examined to provide insight into the characteristics that should be captured in a combat value function. And finally, a heuristic means to derive combat values is proposed.

A. BACKGROUND

1. Combat Value Concept

As discussed in the introduction, the problem of rearming during combat involve objectives which should effectively combine combat objectives and logistics objectives. The measures of effectiveness concerning logistics are inherently easy to define quantitatively, in easily understood units such as time, number of jobs, transportation cost, etc. The measures of effectiveness concerning combat are less easy to define quantitatively, and traditionally a variety of measures have been used to capture the idea of *combat effectiveness* in terms that are both useful and understandable to a rational decision maker. In the prototype model of Chapter III, a very simple combat model was introduced to quantitatively express total combat value as the probability of successful defense of the Battle Group. Each additional lift of ammunition provided a marginal increase in combat value. In that prototype problem, the objective function was an expectation of the total combat value accumulated before the replenishment process terminated due to the arrival of an attack. Another interpretation of the measure of effectiveness is obtained using the terminology of job scheduling theory. As discussed in Chapter IV, the total combat value of lifts completed prior to a raid arrival could also be thought of as the weighted number of jobs completed before their due date, where the due date is the arrival of a raid, and the weights are those same marginal combat values. In this respect, the objective function combines a measure of combat effectiveness with a measure of logistics effectiveness, specifically combat weights (or values) and numbers of jobs completed.

Defining the combat value function in the prototype model of Chapter III as a probability of successful defense was arbitrary. Another utility function could have been used to quantify the value of having some specified number of weapons available to the Battle Group when combat commences.

2. Combat Value Terminology

Suppose the Battle Group is in some depleted weapons state and anticipating combat. And suppose also that the Battle Group Commander could instantaneously (i.e., disregarding logistics considerations) increase the weapons state of the Battle Group by adding one *lift* of ammunition to one combatant. Implicitly, he will choose the lift that has the highest *utility* to him for the ensuing combat. If his decision is based on improving the overall combat capability of the Battle Group, then the particular lift he will choose is the one said to yield the highest total *combat utility*. And since from a single lift , he is getting a marginal improvement in overall combat capability, that lift is the one with the highest *marginal combat utility*. If the concept of utility is quantified, then the measure of utility may be referred to as a *utility value*, or in this case *marginal combat utility value*, or simply *marginal combat value*. For example, if the ammunition loads in all ships in the Battle Group were depleted by approximately the same percentage, then the Battle Group commander may prefer to add one SM2(MR) on an AEGIS cruiser rather than one SM1(MR) on a FFG-7, feeling that the former will have more utility in the ensuing combat.

B. THE PROTOTYPE COMBAT VALUE FUNCTION

The prototype combat model of Chapter III provided a combat value function with the following properties:

1. ***Weapon Effectiveness.*** The marginal combat value of a lift was an increasing function of its single shot probability of kill, which may be arbitrary, but is usually a measure of the combined effectiveness of the weapon round itself and the accuracy of the weapons direction system of the launching platform.

2. ***Platform Effectiveness.*** The marginal combat value of a lift was an increasing function of the probability that the attacker is engageable by defender, which is determined by the defensive function, and hence position, of the defender in the battle group.

3. Diminishing Returns. Total combat value was a concave function of the number of weapons already onboard each receiver. That is, the marginal combat value of an early lift is greater than subsequent lifts. This situation is usually referred to as *diminishing returns.* This property of the model is consistent with intuition concerning Battle Group ammunition. For example, for two ships performing identical missions, the ten missiles it takes to bring one AEGIS cruiser's missile load from 0 to 10 are worth more than the ten it takes to increase another AEGIS cruiser from, say, 70 to 80. That example is straightforward. As another example of the effect of diminishing returns, the Battle Group Commander may prefer to add one SM1(MR) on a FFG-7 that is down to 5 percent of its missiles, rather than one SM2(MR) on a much more capable AEGIS cruiser that is at ninety-five percent. This example reflects the tendency of the Battle Group Commander to want a balance of some sort in how his assigned forces are loaded.

4. Additivity. The marginal combat values of lifts to different receivers added together. This property was a consequence of a simplifying assumption concerning the simple combat model of Chapter IV, which was required to allow the application of the interchange argument in sequencing. It was necessary that the marginal combat values of lifts for each receiver not depend on the states of other receivers. That implied that the total combat value for the battle group was the sum of the combat values of the receivers (i.e., there were no cross terms in the Battle Group combat value function). In that model, the necessary condition was satisfied due to the assumption that attacker engageability by the defenders was mutually exclusive.

C. COMBAT VALUE PRIORITY LIST METHOD

1. Priority List

Consider a Battle Group preparing to enter a combat situation. Conceptually, suppose that there was no ammunition currently aboard any of the ships, and ask the Battle Group Commander to make an expert assessment of the tactical situation and anticipated ammunition demand and to name the first unit of ammunition he would want in the Battle Group and which ship should have it. Next, have him identify the second unit, and so forth. In this manner, a priority list for every unit of ammunition in the Battle Group is generated, based purely on combat considerations. It may be

noted that this priority list can be considered as the *ideal* ammunition loading sequence if logistics factors are disregarded.

The idea of this priority list method is that each item on the list has a higher marginal combat value than the next item on the list. How much higher remains to be discussed. To quantify these differences, a utility scale called *combat potential* is introduced.

2. Combat Potential

Combat potential is used as a measure of how effective a particular unit of ammunition is when available on a particular combatant. Combat potential could perhaps be quantified objectively using such factors as single shot kill probability, weapons system detection, acquisition and maximum engagement range, etc. However, in the following, combat potential is assigned subjectively.

Arbitrarily, combat potential is scaled so that the least capable weapon on the lowest value ship has a combat potential of 1, and other weapon-ship combinations are assigned combat potentials relative to that one. For example, if the least capable weapon on the lowest value ship (in an ASCM environment) is a 5"/38 gun on a frigate, then a lift of such ammunition for that ship has a combat utility potential of 1. If a 5"/54 gun on a destroyer is twice as *good*, then it has a combat utility potential of 2. If CIWS on a destroyer is twice again as *good*, then it has a combat utility potential of 4. And so forth.

3. Calculation of Marginal Combat Values

With the priority list and combat potentials established, marginal combat values of each of the units of ammunition is calculated in a simple way. The marginal combat value of the unit at the bottom of the priority list is set equal to its combat potential. Then proceeding up the priority list, the marginal combat value of each unit is increased above the previous (lower priority) unit by its own combat potential.

4. Properties

The marginal combat values calculated in the manner described above thus capture the basic idea of diminishing returns -- the units of ammunition at the top of the priority list (which conceptually are loaded when the Battle Group is nearly empty) have much higher marginal combat values than the units of ammunition at the bottom of the priority list (which conceptually are loaded when the Battle Group is almost full).

The priority list captures two more of the ideas discussed above. One is the idea of platform effectiveness, which is reflected in the tendency of the Battle Group Commander to give higher priority to those ships which he expects to bear the greatest burden in defending against the next raid. The other idea, which was discussed under diminishing returns, is that the distribution of ammunition in the Battle Group should be somehow balanced among the receivers.

The combat potential captures the idea of weapon round and weapons system effectiveness discussed above.

The additivity property of marginal combat value is inherent in the fact that once it is calculated, the marginal combat value of each unit of ammunition in the Battle Group is fixed, based on pure combat considerations, disregarding the logistics considerations which will play a part in the actual order in which each unit is loaded.

5. Combat Value Priority List Method Outline

The following steps, which will be illustrated in a subsequent example, outline a procedure for establishing marginal combat values by the Combat Value Priority List Method:

Before Battle Groups are formed:

1. Assemble basic data
   - ID of combatants
   - Types of Weapons
   - Capacities

2. Assign Combat Potentials

3. For each combatant (receiver):
   - a. Prioritize every unit of ammunition.
   - b. Sort by receiver priority number.

When Battle Group is formed:

4. For the entire Battle Group:
   - a. Establish integrated BG priorities.

b. Sort by BG priority number.

5. Calculate Marginal Combat Values

6. Example

In this example, for illustration of the heuristic Combat Value Priority List Method, a small Battle Group has three receivers, called *ShipX*, *ShipY*, and *ShipZ*. Within the Battle Group, there are six types of ammunition of interest, called *WepA*, *WepB*, *WepC*, *WepD*, *WepE*, and *WepF*.

The receiving ships and types of weapons each carries are identified in Figure 42, along with the weapons capacities and assigned combat potentials for each.

Receiver	Ammo Type	Capacity	Combat Potential
ShipX	WepA	16	20
ShipX	WepD	2	4
ShipY	WepB	10	10
ShipY	WepD	2	4
ShipY	WepE	2	4
ShipZ	WepC	4	6
ShipZ	WepE	2	4
ShipZ	WepF	2	1

Figure 42. Battle Group Ammunition Summary

For each of the receivers, Figure 43a., b. and c., respectively, lists, by serial number, every individual lift of ammunition carried on that receiver. The serial numbers are simply the order in which that receiver fills up that particular weapon magazine. In the last column of Figure 43a., b., and c., the lifts are numbered with receiver priorities, using the same idea discussed above for the overall Battle Group priority list. This is simply an intermediate step to pre-process ammunition lists for each receiver in preparation for assigning Battle Group priorities. The short list for ShipZ, Figure 43c., pro-

vides a good example of a priority list which is intended to build up a balanced weapons load for that receiver.

Receiver	Ammo Type	Serial Number	Combat Potential	Receiver Priority
ShipX	WepA	1	20	*1*
ShipX	WepA	2	20	*2*
ShipX	WepA	3	20	*3*
ShipX	WepA	4	20	*4*
ShipX	WepA	5	20	*5*
ShipX	WepA	6	20	*6*
ShipX	WepA	7	20	*8*
ShipX	WepA	8	20	*9*
ShipX	WepA	9	20	*10*
ShipX	WepA	10	20	*11*
ShipX	WepA	11	20	*12*
ShipX	WepA	12	20	*14*
ShipX	WepA	13	20	*15*
ShipX	WepA	14	20	*16*
ShipX	WepA	15	20	*17*
ShipX	WepA	16	20	*18*
ShipX	WepD	1	4	*7*
ShipX	WepD	2	4	*13*

Figure 43a. ShipX List by Ser. No. with Rcvr. Pri. Assigned

Receiver	Ammo Type	Serial Number	Combat Potential	Receiver Priority
ShipY	WepB	1	10	*1*
ShipY	WepB	2	10	*2*
ShipY	WepB	3	10	*3*
ShipY	WepB	4	10	*4*
ShipY	WepB	5	10	*7*
ShipY	WepB	6	10	*8*
ShipY	WepB	7	10	*9*
ShipY	WepB	8	10	*10*
ShipY	WepB	9	10	*11*
ShipY	WepB	10	10	*12*
ShipY	WepD	1	4	*5*
ShipY	WepD	2	4	*13*
ShipY	WepF	1	4	*6*
ShipY	WepF	2	4	*14*

Figure 43b. ShipY List by Ser. No. with Rcvr. Pri. Assigned

Receiver	Ammo Type	Serial Number	Combat Potential	Receiver Priority
ShipZ	WepC	1	6	*1*
ShipZ	WepC	2	6	*2*
ShipZ	WepC	3	6	*4*
ShipZ	WepC	4	6	*6*
ShipZ	WepE	1	4	*3*
ShipZ	WepE	2	4	*7*
ShipZ	WepF	1	1	*5*
ShipZ	WepF	2	1	*8*

Figure 43c. ShipZ List by Ser. No. with Rcvr. Pri. Assigned

In next step, the individual receiver lists are sorted by receiver priorities and combined in a Battle Group list as shown in Figure 44. Then Battle Group priorities are assigned.

Receiver	Ammo Type	Serial Number	Combat Potential	Receiver Priority	Group Priority
ShipX	WepA	1	20	1	*1*
ShipX	WepA	2	20	2	*2*
ShipX	WepA	3	20	3	*3*
ShipX	WepA	4	20	4	*4*
ShipX	WepA	5	20	5	*6*
ShipX	WepA	6	20	6	*7*
ShipX	WepD	1	4	7	*9*
ShipX	WepA	7	20	8	*10*
ShipX	WepA	8	20	9	*13*
ShipX	WepA	9	20	10	*16*
ShipX	WepA	10	20	11	*17*
ShipX	WepA	11	20	12	*20*
ShipX	WepD	2	4	13	*21*
ShipX	WepA	12	20	14	*23*
ShipX	WepA	13	20	15	*26*
ShipX	WepA	14	20	16	*27*
ShipX	WepA	15	20	17	*30*
ShipX	WepA	16	20	18	*35*
ShipY	WepB	1	10	1	*5*
ShipY	WepB	2	10	2	*8*
ShipY	WepB	3	10	3	*11*
ShipY	WepB	4	10	4	*14*
ShipY	WepD	1	4	5	*18*
ShipY	WepE	1	4	6	*22*
ShipY	WepB	5	10	7	*24*
ShipY	WepB	6	10	8	*28*
ShipY	WepB	7	10	9	*31*
ShipY	WepB	8	10	10	*32*
ShipY	WepB	9	10	11	*34*
ShipY	WepB	10	10	12	*36*
ShipY	WepD	2	4	13	*38*
ShipY	WepE	2	4	14	*39*
ShipZ	WepC	1	6	1	*12*
ShipZ	WepC	2	6	2	*15*
ShipZ	WepE	1	4	3	*19*
ShipZ	WepC	3	6	4	*25*
ShipZ	WepF	1	1	5	*29*
ShipZ	WepC	4	6	6	*33*
ShipZ	WepE	2	4	7	*37*
ShipZ	WepF	2	1	8	*40*

Figure 44. Group List by Rcvr. & Rcvr. Pri. with Group Pri. Assigned

Finally, the combined group list is sorted by Battle Group priority, and the marginal combat values are calculated, as described above, starting from the bottom of the list. The final list for this example is shown in Figure 45.

D. JUSTIFICATION FOR A HEURISTIC

In general, the combat value of a particular lift depends on many factors. Some of the factors are deterministic and some are stochastic. Examples of deterministic factors are:

Quantity of ammunition currently onboard

Ammunition design characteristics (including: warhead size, type of seeker, type of fuze, etc.)

Weapon System design characteristics (including: type of guidance, type of radars, number of directors, type of launcher, etc.)

Battle Group formation

The stochastic factors fall into several categories. Some stochastic factors are observable and distributional information may be inferred from data. For example:

Weapons system performance (including: maximum effective range, maximum altitude, lethal radius, etc.)

Weapons system degradations (including: failure rate, etc.)

Environmental conditions which affect weapon system performance

Some stochastic factors may not be directly observable, but may be subjectively estimated. For example:

Raid time

Raid size

Raid composition

Raid origin

Threat axis

And finally, some stochastic factors are dominated by such uncertainty that assumptions must be made to permit any type of modeling. For example:

Planned raid tactics (including: grouping of attackers, coordination & sequence of attack (sub, surface, air), target priorities, etc.)

Actual battle dynamics (including sequential decisions made in the face of unforeseen circumstances)

Receiver	Ammo Type	Serial Number	Combat Potential	Receiver Priority	Group Priority	Marginal Combat Value
ShipX	WepA	1	20	1	1	*478*
ShipX	WepA	2	20	2	2	*458*
ShipX	WepA	3	20	3	3	*438*
ShipX	WepA	4	20	4	4	*418*
ShipY	WepB	1	10	1	5	*398*
ShipX	WepA	5	20	5	6	*388*
ShipX	WepA	6	20	6	7	*368*
ShipY	WepB	2	10	2	8	*348*
ShipX	WepD	1	4	7	9	*338*
ShipX	WepA	7	20	8	10	*334*
ShipY	WepB	3	10	3	11	*314*
ShipZ	WepC	1	6	1	12	*304*
ShipX	WepA	8	20	9	13	*298*
ShipY	WepB	4	10	4	14	*278*
ShipZ	WepC	2	6	2	15	*268*
ShipX	WepA	9	20	10	16	*262*
ShipX	WepA	10	20	11	17	*242*
ShipY	WepD	1	4	5	18	*222*
ShipZ	WepE	1	4	3	19	*218*
ShipX	WepA	11	20	12	20	*214*
ShipX	WepD	2	4	13	21	*194*
ShipY	WepF	1	4	6	22	*190*
ShipX	WepA	12	20	14	23	*186*
ShipY	WepB	5	10	7	24	*166*
ShipZ	WepC	3	6	4	25	*156*
ShipX	WepA	13	20	15	26	*150*
ShipX	WepA	14	20	16	27	*130*
ShipY	WepB	6	10	8	28	*110*
ShipZ	WepF	1	1	5	29	*100*
ShipX	WepA	15	20	17	30	*99*
ShipY	WepB	7	10	9	31	*79*
ShipY	WepB	8	10	10	32	*69*
ShipZ	WepC	4	6	6	33	*59*
ShipY	WepB	9	10	11	34	*53*
ShipX	WepA	16	20	18	35	*43*
ShipY	WepB	10	10	12	36	*23*
ShipZ	WepE	2	4	7	37	*13*
ShipY	WepD	2	4	13	38	*9*
ShipY	WepE	2	4	14	39	*5*
ShipZ	WepF	2	1	8	40	*1*

Figure 45. Group List by Group Pri. with Marg. Combat Values Calculated

The word "scenario" combines elements of many of these factors. Invariably, scenarios are postulated which includes assumptions concerning raid tactics, roughly fixing raid variables which could be estimated (such as raid size: one, few, many), fixing other raid variables (such as threat axis and attack plan), and ignoring battle dynamics.

Building general combat models quickly leads to very high scenario dependency to account for important deterministic and stochastic variables, and which by their inherent complexity must use more and more assumptions. Because realistic analytic combat models become too complex and scenario dependent, simulation and wargaming are invariably resorted to for complex combat scenarios. However, a simplified analytic combat model which may not capture all the fine-grained detail of a real problem may by its simplicity provide important insight into the general behavior of a process being modeled.

Thus the approach used in this study has been to use the highly simplified combat model of Chapter III to analytically identify the properties of a combat value function, and then use a heuristic method to capture more realistic considerations.

APPENDIX C. PERT REPRESENTATION OF TRANSFER TIMES

This appendix contains the details of how a replenishment transfer may be represented as a PERT type system; see, for example, Elmaghraby [Ref. 23]. For simplicity, this discussion will consider the process of VERTREP of a single commodity from a delivery ship to a single receiver by one helicopter. Extension to multiple receivers, weapons, and helicopters is simply a matter of additional subscripts on several of the variables. All times are deterministic.

Following PERT terminology, the word *activity* will be used for the various portions of the replenishment process which take place over an interval of time, including breakout, helicopter travel, and strikedown. The word *event* will be used to mark the instant in time at which an activity starts or finishes. For clarity, multi-letter variable names will be used in this description. Variables which represent event times will be prefixed with E and variables which represent the time it takes to conduct an activity will be prefixed with T. Let the index i denote the sequence in which lifts are transferred.

A. ACTIVITY TIMES

Let the following variables denote the corresponding activity times:

- Tb_i Time it takes the delivery ship to process (break out) the i^{th} lift.
- Tr_i Time it takes the receiving ship to process the i^{th} lift.
- Tg_i Time it takes the helicopter to get the i^{th} lift.
- Td_i Time it takes the helicopter to drop off the i^{th} lift.
- Tvo_i Time it takes the helicopter to travel out from the delivery ship to the receiver with the i^{th} lift.
- Tvi_i Time it takes the helicopter to travel in from the receiver to the delivery ship after dropping the i^{th} lift.

Delivery ship processing activity time includes the time it takes to remove items from storage, package items into a ready-for-transfer lift, and stage the lift for pickup by the helicopter.

Receiver processing time includes the time to un-package the lift, and make items from the lift ready for use. Collectively, these activities are part of the total time it takes a receiver to complete strikedown. The other part, which is not included in Tr_i is any time that the lift spends in a strikedown queue.

The time it takes a helicopter to get a lift includes the fixed time for the helicopter to maneuver into pickup position, pick up the staged lift from the delivery ship, and start moving towards the delivery ship. This *fixed* time is exclusive of variable flight time flying from the delivery ship to the receiver.

The time it takes a helicopter to drop off a lift includes the fixed time for the helicopter to maneuver into drop off position and actually make the drop. This also is exclusive of variable flight time.

The helicopter travel times are variable due to the relative speeds of the receiving ship, delivery ship, and helicopter, and the distance the helicopter must travel each direction.

B. EVENT TIMES

Let the following variables denote the corresponding event times:

Eb_i	Event time when the delivery ship has completed breakout of the i^{th} lift.
Ed_i	Event time when the helicopter is dispatched with the i^{th} lift.
Eh_i	Event time when the helicopter has dropped off the i^{th} lift.
Es_i	Event time when the receiver has completed strikedown of the i^{th} lift.
Er_i	Event time when the helicopter has returned from dropping off the i^{th} lift.

1. Recursive Calculation of Event Times

The event times may be calculated recursively. The event which marks the time when the delivery ship has completed breakout of current lift is the length of the current lift breakout activity time added to the time when the previous lift breakout was complete, as follows.

$$Eb_i = Eb_{i-1} + Tb_i$$

A helicopter may be dispatched at the latter of its return from a previous lift or breakout completion of the current lift, plus the fixed time it takes the helicopter to pick up the lift, as follows:

$$Ed_i = Tg_i + \max\{Eb_i, Er_{i-1}\}$$

The time at which a lift is dropped off at the receiver is the sum of the time of dispatch plus variable flight time to the receiver plus the fixed time to drop off the lift, as follows:

$$Eh_i = Ed_i + Tvo_i + Th_i$$

The event which marks the time of strikedown completion is the length of the current lift strikedown activity time added to the latter of the previous strikedown completion event or the current lift drop off time, as follows:

$$Es_i = Tr_i + \max\{Eh_i, Es_{i-1}\}$$

The event which marks the helicopter's return from the current round-trip and readiness to pick up the next lift is the variable flight time returning added to the event time when the current lift was dropped off, as follows:

$$Er_i = Eh_i + Tvi_i$$

2. PERT Diagram Representation

A segment of a PERT diagram representation corresponding to the computations given above is shown in Figure 46. The large circles represent the VERTREP events, and the solid arrows represent the VERTREP activities. The small circles and the dashed arrows are dummy events and activities which are used to represent precedence on a PERT network

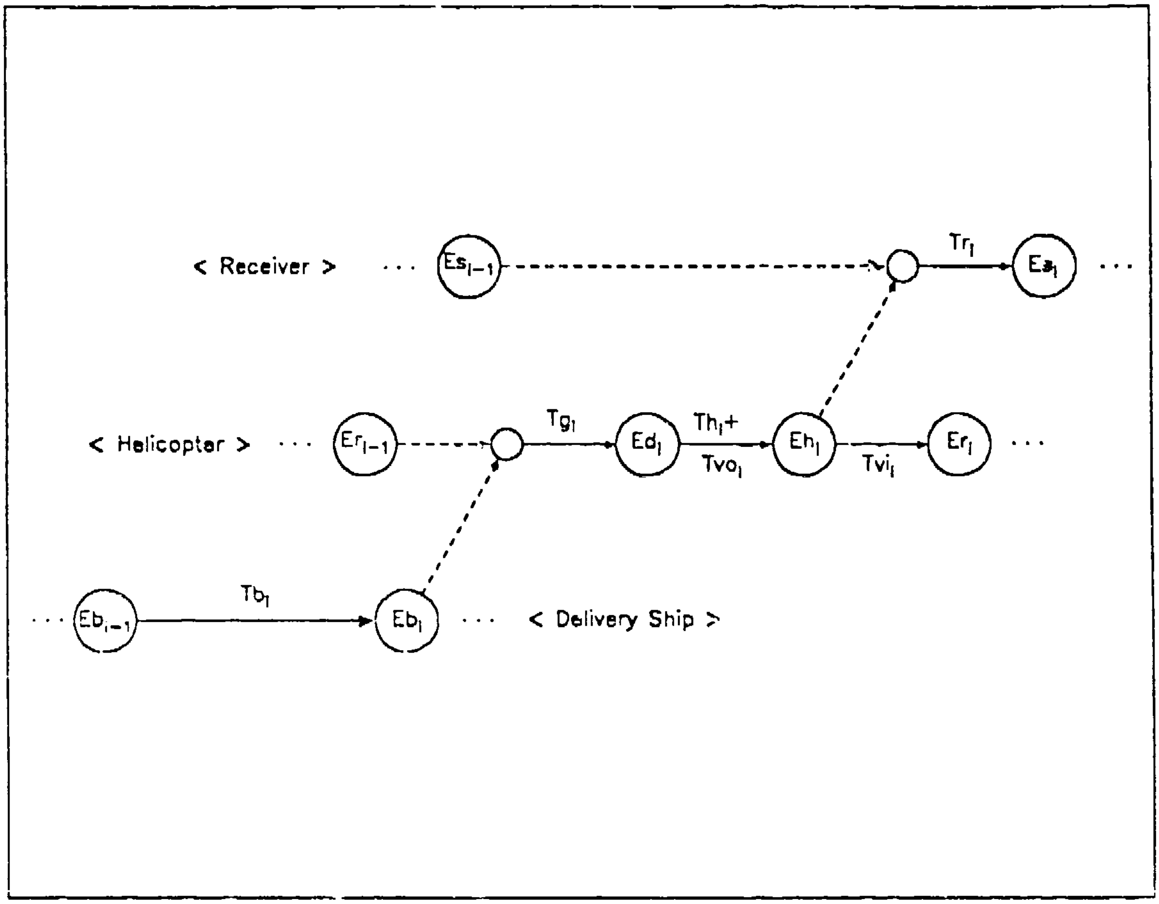


Figure 46. PERT Diagram Segment

3. Initialization

If all replenishment activity starts at time zero with no lifts waiting to be processed, event times are initialized as follows:

$$Eb_1 = Tb_1$$

$$Ed_1 = Tg_1 + Eb_1$$

$$Eh_1 = Ed_1 + Tvo_1 + Th_1$$

$$Es_1 = Tr_1 + Eh_1$$

$$Er_1 = Eh_1 + Tvi_1$$

APPENDIX D. BATTLE GROUP VERTREP EXAMPLE

This appendix contains the tables pertaining to the example of a Battle Group combat VERTREP problem discussed in Chapter III.

Table 6. BATTLE GROUP AMMUNITION SUMMARY (COMBAT VALUE INPUT)

Receiver	Ammo Type	Capacity	Combat Potential
Ship1	WepA	40	20
Ship1	WepD	4	4
Ship1	WepE	4	4
Ship1	WepF	20	2
Ship2	WepB	20	10
Ship2	WepD	2	4
Ship2	WepE	2	4
Ship2	WepF	10	2
Ship3	WepC	8	6
Ship3	WepD	2	4
Ship3	WepE	2	4
Ship3	WepF	20	2
Ship4	WepC	4	6
Ship4	WepD	1	4
Ship4	WepE	2	4
Ship4	WepG	10	1

Table 7. SHIP1 LIST BY SER. NO. WITH RCVR. PRI. ASSIGNED

Receiver	Ammo Type	Serial Number	Combat Potential	Receiver Priority
Ship1	WepA	1	20	1
Ship1	WepA	2	20	2
Ship1	WepA	3	20	3
Ship1	WepA	4	20	4
Ship1	WepA	5	20	5
Ship1	WepA	6	20	6
Ship1	WepA	7	20	7
Ship1	WepA	8	20	8
Ship1	WepA	9	20	11
Ship1	WepA	10	20	12
Ship1	WepA	11	20	13
Ship1	WepA	12	20	14
Ship1	WepA	13	20	15
Ship1	WepA	14	20	16
Ship1	WepA	15	20	17
Ship1	WepA	16	20	18
Ship1	WepA	17	20	21
Ship1	WepA	18	20	22
Ship1	WepA	19	20	23
Ship1	WepA	20	20	24
Ship1	WepA	21	20	26
Ship1	WepA	22	20	27
Ship1	WepA	23	20	28
Ship1	WepA	24	20	29
Ship1	WepA	25	20	30
Ship1	WepA	26	20	31
Ship1	WepA	27	20	32
Ship1	WepA	28	20	33
Ship1	WepA	29	20	34
Ship1	WepA	30	20	35
Ship1	WepA	31	20	39
Ship1	WepA	32	20	40
Ship1	WepA	33	20	41
Ship1	WepA	34	20	42
Ship1	WepA	35	20	43
Ship1	WepA	36	20	44
Ship1	WepA	37	20	45
Ship1	WepA	38	20	46
Ship1	WepA	39	20	47
Ship1	WepA	40	20	48
Ship1	WepD	1	4	9
Ship1	WepD	2	4	19
Ship1	WepD	3	4	36
Ship1	WepD	4	4	49
Ship1	WepE	1	4	10
Ship1	WepE	2	4	20
Ship1	WepE	3	4	37

Ship1	WepE	4	4	50
Ship1	WepF	1	2	25
Ship1	WepF	2	2	38
Ship1	WepF	3	2	51
Ship1	WepF	4	2	52
Ship1	WepF	5	2	53
Ship1	WepF	6	2	54
Ship1	WepF	7	2	55
Ship1	WepF	8	2	56
Ship1	WepF	9	2	57
Ship1	WepF	10	2	58
Ship1	WepF	11	2	59
Ship1	WepF	12	2	60
Ship1	WepF	13	2	61
Ship1	WepF	14	2	62
Ship1	WepF	15	2	63
Ship1	WepF	16	2	64
Ship1	WepF	17	2	65
Ship1	WepF	18	2	66
Ship1	WepF	19	2	67
Ship1	WepF	20	2	68

Table 8. SHIP2 LIST BY SER. NO. WITH RCVR. PRI. ASSIGNED

Receiver	Ammo Type	Serial Number	Combat Potential	Receiver Priority
Ship2	WepB	1	10	1
Ship2	WepB	2	10	2
Ship2	WepB	3	10	3
Ship2	WepB	4	10	4
Ship2	WepB	5	10	7
Ship2	WepB	6	10	8
Ship2	WepB	7	10	9
Ship2	WepB	8	10	10
Ship2	WepB	9	10	11
Ship2	WepB	10	10	12
Ship2	WepB	11	10	14
Ship2	WepB	12	10	15
Ship2	WepB	13	10	17
Ship2	WepB	14	10	18
Ship2	WepB	15	10	20
Ship2	WepB	16	10	21
Ship2	WepB	17	10	24
Ship2	WepB	18	10	25
Ship2	WepB	19	10	27
Ship2	WepB	20	10	28
Ship2	WepD	1	4	5
Ship2	WepD	2	4	22
Ship2	WepE	1	4	6
Ship2	WepE	2	4	26
Ship2	WepF	1	2	13
Ship2	WepF	2	2	16
Ship2	WepF	3	2	19
Ship2	WepF	4	2	23
Ship2	WepF	5	2	29
Ship2	WepF	6	2	30
Ship2	WepF	7	2	31
Ship2	WepF	8	2	32
Ship2	WepF	9	2	33
Ship2	WepF	10	2	34

Table 9. SHIP3 LIST BY SER. NO. WITH RCVR. PRI. ASSIGNED

Receiver	Ammo Type	Serial Number	Combat Potential	Receiver Priority
Ship3	WepC	1	6	1
Ship3	WepC	2	6	2
Ship3	WepC	3	6	4
Ship3	WepC	4	6	5
Ship3	WepC	5	6	7
Ship3	WepC	6	6	8
Ship3	WepC	7	6	10
Ship3	WepC	8	6	12
Ship3	WepD	1	4	3
Ship3	WepD	2	4	13
Ship3	WepE	1	4	6
Ship3	WepE	2	4	14
Ship3	WepF	1	2	9
Ship3	WepF	2	2	11
Ship3	WepF	3	2	15
Ship3	WepF	4	2	16
Ship3	WepF	5	2	17
Ship3	WepF	6	2	18
Ship3	WepF	7	2	19
Ship3	WepF	8	2	20
Ship3	WepF	9	2	21
Ship3	WepF	10	[illegible]	22
Ship3	WepF	11	2	23
Ship3	WepF	12	2	24
Ship3	WepF	13	2	25
Ship3	WepF	14	2	26
Ship3	WepF	15	2	27
Ship3	WepF	16	2	28
Ship3	WepF	17	2	29
Ship3	WepF	18	2	30
Ship3	WepF	19	2	31
Ship3	WepF	20	2	32

Table 10. SHIP4 LIST BY SER. NO. WITH RCVR. PRI. ASSIGNED

Receiver	Ammo Type	Serial Number	Combat Potential	Receiver Priority
Ship4	WepC	1	6	1
Ship4	WepC	2	6	2
Ship4	WepC	3	6	5
Ship4	WepC	4	6	7
Ship4	WepD	1	4	3
Ship4	WepE	1	4	4
Ship4	WepE	2	4	8
Ship4	WepG	1	1	6
Ship4	WepG	2	1	9
Ship4	WepG	3	1	10
Ship4	WepG	4	1	11
Ship4	WepG	5	1	12
Ship4	WepG	6	1	13
Ship4	WepG	7	1	14
Ship4	WepG	8	1	15
Ship4	WepG	9	1	16
Ship4	WepG	10	1	17

Table 11. GROUP LIST BY RCVR. & RCVR. PRI. WITH GROUP PRI. ASSIGNED

Receiver	Ammo Type	Serial Number	Combat Potential	Receiver Priority	Group Priority
Ship1	WepA	20	1	1	1
Ship1	WepA	20	2	2	2
Ship1	WepA	20	3	3	3
Ship1	WepA	20	4	4	4
Ship1	WepA	20	5	5	6
Ship1	WepA	20	6	6	7
Ship1	WepA	20	7	7	8
Ship1	WepA	20	8	8	9
Ship1	WepD	4	1	9	11
Ship1	WepE	4	1	10	23
Ship1	WepA	20	9	11	15
Ship1	WepA	20	10	12	16
Ship1	WepA	20	11	13	20
Ship1	WepA	20	12	14	21
Ship1	WepA	20	13	15	27
Ship1	WepA	20	14	16	28
Ship1	WepA	20	15	17	34
Ship1	WepA	20	16	18	35
Ship1	WepD	4	2	19	38
Ship1	WepE	4	2	20	42
Ship1	WepA	20	17	21	44
Ship1	WepA	20	18	22	45
Ship1	WepA	20	19	23	48
Ship1	WepA	20	20	24	49
Ship1	WepF	2	1	25	53
Ship1	WepA	20	21	26	54
Ship1	WepA	20	22	27	57
Ship1	WepA	20	23	28	58
Ship1	WepA	20	24	29	60
Ship1	WepA	20	25	30	61
Ship1	WepA	20	26	31	65
Ship1	WepA	20	27	32	66
Ship1	WepA	20	28	33	67
Ship1	WepA	20	29	34	70
Ship1	WepA	20	30	35	72
Ship1	WepD	4	3	36	73
Ship1	WepE	4	3	37	74
Ship1	WepF	2	2	38	77
Ship1	WepA	20	31	39	80
Ship1	WepA	20	32	40	81
Ship1	WepA	20	33	41	83
Ship1	WepA	20	34	42	85
Ship1	WepA	20	35	43	86
Ship1	WepA	20	36	44	88
Ship1	WepA	20	37	45	89
Ship1	WepA	20	38	46	91

Ship1	WepA	20	39	47	93
Ship1	WepA	20	40	48	95
Ship1	WepD	4	4	49	98
Ship1	WepE	4	4	50	100
Ship1	WepF	2	3	51	102
Ship1	WepF	2	4	52	105
Ship1	WepF	2	5	53	106
Ship1	WepF	2	6	54	110
Ship1	WepF	2	7	55	112
Ship1	WepF	2	8	56	115
Ship1	WepF	2	9	57	118
Ship1	WepF	2	10	58	121
Ship1	WepF	2	11	59	124
Ship1	WepF	2	12	60	126
Ship1	WepF	2	13	61	129
Ship1	WepF	2	14	62	132
Ship1	WepF	2	15	63	136
Ship1	WepF	2	16	64	138
Ship1	WepF	2	17	65	140
Ship1	WepF	2	18	66	142
Ship1	WepF	2	19	67	144
Ship1	WepF	2	20	68	148
Ship2	WepB	10	1	1	5
Ship2	WepB	10	2	2	10
Ship2	WepB	10	3	3	17
Ship2	WepB	10	4	4	22
Ship2	WepD	4	1	5	12
Ship2	WepE	4	1	6	24
Ship2	WepB	10	5	7	29
Ship2	WepB	10	6	8	30
Ship2	WepB	10	7	9	36
Ship2	WepB	10	8	10	39
Ship2	WepB	10	9	11	43
Ship2	WepB	10	10	12	50
Ship2	WepF	2	1	13	55
Ship2	WepB	10	11	14	59
Ship2	WepB	10	12	15	62
Ship2	WepF	2	2	16	68
Ship2	WepB	10	13	17	75
Ship2	WepB	10	14	18	78
Ship2	WepF	2	3	19	82
Ship2	WepB	10	15	20	84
Ship2	WepB	10	16	21	87
Ship2	WepD	4	2	22	90
Ship2	WepF	2	4	23	92
Ship2	WepB	10	17	24	94
Ship2	WepB	10	18	25	96
Ship2	WepE	4	2	26	97
Ship2	WepB	10	19	27	99
Ship2	WepB	10	20	28	101
Ship2	WepF	2	5	29	114
Ship2	WepF	2	6	30	120
Ship2	WepF	2	7	31	128
Ship2	WepF	2	8	32	134
Ship2	WepF	2	9	33	145

Ship2	WepF	2	10	34	149
Ship3	WepC	6	1	1	13
Ship3	WepC	6	2	2	18
Ship3	WepD	4	1	3	25
Ship3	WepC	6	3	4	31
Ship3	WepC	6	4	5	32
Ship3	WepE	4	1	6	37
Ship3	WepC	6	5	7	40
Ship3	WepC	6	6	8	46
Ship3	WepF	2	1	9	51
Ship3	WepC	6	7	10	56
Ship3	WepF	2	2	11	63
Ship3	WepC	6	8	12	69
Ship3	WepD	4	2	13	71
Ship3	WepE	4	2	14	76
Ship3	WepF	2	3	15	103
Ship3	WepF	2	4	16	107
Ship3	WepF	2	5	17	109
Ship3	WepF	2	6	18	111
Ship3	WepF	2	7	19	113
Ship3	WepF	2	8	20	116
Ship3	WepF	2	9	21	119
Ship3	WepF	2	10	22	122
Ship3	WepF	2	11	23	125
Ship3	WepF	2	12	24	127
Ship3	WepF	2	13	25	130
Ship3	WepF	2	14	26	133
Ship3	WepF	2	15	27	137
Ship3	WepF	2	16	28	139
Ship3	WepF	2	17	29	141
Ship3	WepF	2	18	30	143
Ship3	WepF	2	19	31	146
Ship3	WepF	2	20	32	150
Ship4	WepC	6	1	1	14
Ship4	WepC	6	2	2	19
Ship4	WepD	4	1	3	26
Ship4	WepE	4	1	4	33
Ship4	WepC	6	3	5	41
Ship4	WepG	1	1	6	47
Ship4	WepC	6	4	7	52
Ship4	WepE	4	2	8	64
Ship4	WepG	1	2	9	79
Ship4	WepG	1	3	10	104
Ship4	WepG	1	4	11	108
Ship4	WepG	1	5	12	117
Ship4	WepG	1	6	13	123
Ship4	WepG	1	7	14	131
Ship4	WepG	1	8	15	135
Ship4	WepG	1	9	16	147
Ship4	WepG	1	10	17	151

Table 12. GROUP LIST BY GROUP PRI. WITH MARG. COMBAT VALUES CALCULATED

Receiver	Ammo Type	Serial Number	Combat Potential	Receiver Priority	Group Priority	Marg. Combat Value
Ship1	WepA	20	1	1	1	1258
Ship1	WepA	20	2	2	2	1238
Ship1	WepA	20	3	3	3	1218
Ship1	WepA	20	4	4	4	1198
Ship2	WepB	10	1	1	5	1178
Ship1	WepA	20	5	5	6	1168
Ship1	WepA	20	6	6	7	1148
Ship1	WepA	20	7	7	8	1128
Ship1	WepA	20	8	8	9	1108
Ship2	WepB	10	2	2	10	1088
Ship1	WepD	4	1	9	11	1078
Ship2	WepD	4	1	5	12	1074
Ship3	WepC	6	1	1	13	1070
Ship4	WepC	6	1	1	14	1064
Ship1	WepA	20	9	11	15	1058
Ship1	WepA	20	10	12	16	1038
Ship2	WepB	10	3	3	17	1018
Ship3	WepC	6	2	2	18	1008
Ship4	WepC	6	2	2	19	1002
Ship1	WepA	20	11	13	20	996
Ship1	WepA	20	12	14	21	976
Ship2	WepB	10	4	4	22	956
Ship1	WepE	4	1	10	23	946
Ship2	WepE	4	1	6	24	942
Ship3	WepD	4	1	3	25	938
Ship4	WepD	4	1	3	26	934
Ship1	WepA	20	13	15	27	930
Ship1	WepA	20	14	16	28	910
Ship2	WepB	10	5	7	29	890
Ship2	WepB	10	6	8	30	880
Ship3	WepC	6	3	4	31	870
Ship3	WepC	6	4	5	32	864
Ship4	WepE	4	1	4	33	858
Ship1	WepA	20	15	17	34	854
Ship1	WepA	20	16	18	35	834
Ship2	WepB	10	7	9	36	814
Ship3	WepE	4	1	6	37	804
Ship1	WepD	4	2	19	38	800
Ship2	WepB	10	8	10	39	796
Ship3	WepC	6	5	7	40	786
Ship4	WepC	6	3	5	41	780
Ship1	WepE	4	2	20	42	774
Ship2	WepB	10	9	11	43	770
Ship1	WepA	20	17	21	44	760
Ship1	WepA	20	18	22	45	740

Ship3	WepC	6	6	8	46	720
Ship4	WepG	1	1	6	47	714
Ship1	WepA	20	19	23	48	713
Ship1	WepA	20	20	24	49	693
Ship2	WepB	10	10	12	50	673
Ship3	WepF	2	1	9	51	663
Ship4	WepC	6	4	7	52	661
Ship1	WepF	2	1	25	53	655
Ship1	WepA	20	21	26	54	633
Ship2	WepF	2	1	13	55	633
Ship3	WepC	6	7	10	56	631
Ship1	WepA	20	22	27	57	625
Ship1	WepA	20	23	28	58	605
Ship2	WepB	10	11	14	59	585
Ship1	WepA	20	24	29	60	575
Ship1	WepA	20	25	30	61	555
Ship2	WepB	10	12	15	62	535
Ship3	WepF	2	2	11	63	525
Ship4	WepE	4	2	8	64	523
Ship1	WepA	20	26	31	65	519
Ship1	WepA	20	27	32	66	499
Ship1	WepA	20	28	33	67	479
Ship2	WepF	2	2	16	68	459
Ship3	WepC	6	8	12	69	457
Ship1	WepA	20	29	34	70	451
Ship3	WepD	4	2	13	71	431
Ship1	WepA	20	30	35	72	427
Ship1	WepD	4	3	36	73	407
Ship1	WepE	4	3	37	74	403
Ship2	WepB	10	13	17	75	399
Ship3	WepE	4	2	14	76	389
Ship1	WepF	2	2	38	77	385
Ship2	WepB	10	14	18	78	383
Ship4	WepG	1	2	9	79	373
Ship1	WepA	20	31	39	80	372
Ship1	WepA	20	32	40	81	352
Ship2	WepF	2	3	19	82	332
Ship1	WepA	20	33	41	83	330
Ship2	WepB	10	15	20	84	310
Ship1	WepA	20	34	42	85	300
Ship1	WepA	20	35	43	86	280
Ship2	WepB	10	16	21	87	260
Ship1	WepA	20	36	44	88	250
Ship1	WepA	20	37	45	89	230
Ship2	WepD	4	2	22	90	210
Ship1	WepA	20	38	46	91	206
Ship2	WepF	2	4	23	92	186
Ship1	WepA	20	39	47	93	184
Ship2	WepB	10	17	24	94	164
Ship1	WepA	20	40	48	95	154
Ship2	WepB	10	18	25	96	134
Ship2	WepE	4	2	26	97	124
Ship1	WepD	4	4	49	98	120
Ship2	WepB	10	19	27	99	116
Ship1	WepE	4	4	50	100	106
Ship2	WepB	10	20	28	101	102

Ship1	WepF	2	3	51	102	92
Ship3	WepF	2	3	15	103	90
Ship4	WepG	1	3	10	104	88
Ship1	WepF	2	4	52	105	87
Ship1	WepF	2	5	53	106	85
Ship3	WepF	2	4	16	107	83
Ship4	WepG	1	4	11	108	81
Ship3	WepF	2	5	17	109	80
Ship1	WepF	2	6	54	110	78
Ship3	WepF	2	6	18	111	76
Ship1	WepF	2	7	55	112	74
Ship3	WepF	2	7	19	113	72
Ship2	WepF	2	5	29	114	70
Ship1	WepF	2	8	56	115	68
Ship3	WepF	2	8	20	116	66
Ship4	WepG	1	5	12	117	64
Ship1	WepF	2	9	57	118	63
Ship3	WepF	2	9	21	119	61
Ship2	WepF	2	6	30	120	59
Ship1	WepF	2	10	58	121	57
Ship3	WepF	2	10	22	122	55
Ship4	WepG	1	6	13	123	53
Ship1	WepF	2	11	59	124	52
Ship3	WepF	2	11	23	125	50
Ship1	WepF	2	12	60	126	48
Ship3	WepF	2	12	24	127	46
Ship2	WepF	2	7	31	128	44
Ship1	WepF	2	13	61	129	42
Ship3	WepF	2	13	25	130	40
Ship4	WepG	1	7	14	131	38
Ship1	WepF	2	14	62	132	37
Ship3	WepF	2	14	26	133	35
Ship2	WepF	2	8	32	134	33
Ship4	WepG	1	8	15	135	31
Ship1	WepF	2	15	63	136	30
Ship3	WepF	2	15	27	137	28
Ship1	WepF	2	16	64	138	26
Ship3	WepF	2	16	28	139	24
Ship1	WepF	2	17	65	140	22
Ship3	WepF	2	17	29	141	20
Ship1	WepF	2	18	66	142	18
Ship3	WepF	2	18	30	143	16
Ship1	WepF	2	19	67	144	14
Ship2	WepF	2	9	33	145	12
Ship3	WepF	2	19	31	146	10
Ship4	WepG	1	9	16	147	8
Ship1	WepF	2	20	68	148	7
Ship2	WepF	2	10	34	149	5
Ship3	WepF	2	20	32	150	3
Ship4	WepG	1	10	17	151	1

Table 13. RECEIVER AMMUNITION REQUESTS (LOGISTICS INPUT)

Receiver	Ammo Type	Strike-down Time	Capacity	Weapon State	Lifts Req.
Ship1	WepA	0.50	40	10	30
Ship1	WepD	0.10	4	1	3
Ship1	WepE	0.20	4	2	3
Ship1	WepF	0.15	20	15	5
Ship2	WepB	0.25	20	5	15
Ship2	WepD	0.10	2	1	1
Ship2	WepE	0.20	2	1	1
Ship2	WepF	0.15	10	4	6
Ship3	WepC	0.20	8	2	6
Ship3	WepD	0.10	2	0	2
Ship3	WepE	0.20	2	0	2
Ship3	WepF	0.15	20	10	10
Ship4	WepC	0.20	4	0	4
Ship4	WepD	0.10	1	0	1
Ship4	WepE	0.20	2	0	2
Ship4	WepG	0.15	10	4	6

Table 14. AMMUNITION DELIVERY DATA (LOGISTICS INPUT)

Ammo Type	Break-out Time	Helo Pickup Time	Helo Dropoff Time
WepA	.16	.03	.02
WepB	.15	.03	.02
WepC	.12	.03	.02
WepD	.10	.03	.02
WepE	.08	.03	.02
WepF	.12	.03	.02
WepG	.10	.03	.02

Table 15. BATTLE GROUP MANEUVERING DATA

Receiver	Ship1	Ship2	Ship3	Ship4
Initial Range	30	5	15	3
Relative Closing Speed	30	20	25	25
Final Station	5	30	3	15
Required on-station Time	8	8	8	8
Relative Opening Speed	20	30	25	25
Helo Relative Delivery Speed	110	90	105	95
Helo Relative Return Speed	90	110	95	105

Table 16. INITIAL VERTREP SCHEDULE - LWCV HEURISTIC

Sequence Number	Dispatch Time	Receiver	Weapon	Marginal Combat Value	Strike-down Compl.
1	.11	Ship4	WepE	858	.33
2	.21	Ship4	WepD	934	.33
3	.33	Ship4	WepC	1064	.55
4	.45	Ship4	WepC	1002	.75
5	.55	Ship3	WepD	938	.68
6	.63	Ship3	WepE	804	.85
7	.75	Ship3	WepC	870	.97
8	.87	Ship3	WepC	864	1.17
9	.95	Ship1	WepE	774	1.18
10	1.05	Ship1	WepD	800	1.17
11	1.13	Ship4	WepE	523	1.35
12	1.25	Ship3	WepC	786	1.47
13	1.37	Ship4	WepC	780	1.59
14	1.49	Ship3	WepC	720	1.71
15	1.64	Ship2	WepB	880	1.91
16	1.80	Ship1	WepA	996	2.32
17	1.92	Ship4	WepC	661	2.14
18	2.07	Ship2	WepB	814	2.34
19	2.23	Ship1	WepA	976	2.82
20	2.35	Ship3	WepC	631	2.57
21	2.50	Ship2	WepB	796	2.77
22	2.66	Ship1	WepA	930	3.32
23	2.74	Ship1	WepE	403	2.96
24	2.89	Ship2	WepB	770	3.16
25	3.05	Ship1	WepA	910	3.82
26	3.13	Ship3	WepE	369	3.35
27	3.23	Ship3	WepD	431	3.35
28	3.39	Ship1	WepA	854	4.32
29	3.54	Ship2	WepB	673	3.81
30	3.64	Ship1	WepD	407	3.76
31	3.80	Ship1	WepA	834	4.82
32	3.95	Ship2	WepB	585	4.22
33	4.07	Ship3	WepC	457	4.29
34	4.23	Ship1	WepA	760	5.32
35	4.38	Ship2	WepB	535	4.65
36	4.54	Ship1	WepA	740	5.82
37	4.70	Ship1	WepA	713	6.32
38	4.86	Ship1	WepA	693	6.82
39	5.01	Ship2	WepB	399	5.28
40	5.17	Ship1	WepA	653	7.32
41	5.32	Ship2	WepB	383	5.59
42	5.48	Ship1	WepA	625	7.82
43	5.58	Ship2	WepD	210	5.70
44	5.74	Ship1	WepA	605	8.32
45	5.89	Ship2	WepB	310	6.16
46	6.05	Ship1	WepA	575	8.82

47	6.20	Ship2	WepB	260	6.47
48	6.36	Ship1	WepA	555	9.32
49	6.44	Ship2	WepE	124	6.66
50	6.60	Ship1	WepA	519	9.82
51	6.76	Ship1	WepA	499	10.32
52	6.84	Ship1	WepE	106	7.06
53	6.94	Ship1	WepD	120	7.06
54	7.10	Ship1	WepA	479	10.82
55	7.26	Ship1	WepA	451	11.32
56	7.42	Ship1	WepA	427	11.82
57	7.58	Ship1	WepA	372	12.32
58	7.74	Ship1	WepA	352	12.82
59	7.86	Ship3	WepF	50	8.03
60	8.02	Ship1	WepA	330	13.32
61	8.18	Ship1	WepA	300	13.82
62	8.33	Ship3	WepF	46	8.53
63	8.46	Ship1	WepA	280	14.32
64	8.61	Ship3	WepF	40	8.81
65	8.74	Ship1	WepA	250	14.82
66	8.89	Ship3	WepF	35	9.09
67	9.02	Ship1	WepA	230	15.32
68	9.18	Ship1	WepA	206	15.82
69	9.33	Ship3	WepF	28	9.53
70	9.45	Ship2	WepB	164	10.05
71	10.11	Ship1	WepA	184	16.32
72	10.26	Ship3	WepF	24	10.46
73	10.37	Ship1	WepA	154	16.82
74	10.52	Ship2	WepB	134	11.12
75	11.17	Ship4	WepG	64	11.50
76	11.53	Ship3	WepF	20	11.72
77	11.64	Ship1	WepF	26	11.85
78	11.79	Ship2	WepB	116	12.39
79	12.44	Ship4	WepG	53	12.77
80	12.79	Ship2	WepB	102	13.40
81	13.45	Ship1	WepF	22	13.66
82	13.60	Ship3	WepF	16	13.80
83	13.71	Ship1	WepF	18	13.93
84	13.86	Ship4	WepG	38	14.19
85	14.21	Ship2	WepF	70	14.72
86	14.87	Ship1	WepF	14	15.08
87	15.02	Ship3	WepF	10	15.22
88	15.13	Ship2	WepF	59	15.63
89	15.79	Ship4	WepG	31	16.11
90	16.14	Ship2	WepF	44	16.64
91	16.79	Ship2	WepF	33	17.30
92	17.45	Ship1	WepF	7	17.66
93	17.60	Ship3	WepF	3	17.80
94	17.71	Ship4	WepG	8	18.04
95	18.06	Ship2	WepF	12	18.56
96	18.72	Ship2	WepF	5	19.22
97	19.37	Ship4	WepG	1	19.70
RTN:	19.69	EXPECTED COMBAT VALUE = 17336.52			

Table 17. 2-OPT VERTREP SCHEDULE - LOCAL NEIGHBORHOOD SEARCH

Sequence Number	Dispatch Time	Receiver	Weapon	Marginal Combat Value	Strike-down Compl.
1	.11	Ship4	WepE	858	.33
2	.21	Ship4	WepD	934	.33
3	.33	Ship4	WepC	1064	.55
4	.41	Ship3	WepE	804	.67
5	.53	Ship4	WepC	1002	.75
6	.63	Ship3	WepD	938	.75
7	.75	Ship3	WepC	870	.97
8	.91	Ship1	WepA	996	1.45
9	1.00	Ship1	WepE	774	1.22
10	1.09	Ship1	WepD	800	1.21
11	1.21	Ship3	WepC	864	1.43
12	1.29	Ship4	WepE	523	1.51
13	1.45	Ship1	WepA	976	1.97
14	1.57	Ship3	WepC	786	1.79
15	1.72	Ship2	WepB	880	1.99
16	1.88	Ship1	WepA	930	2.47
17	2.00	Ship4	WepC	780	2.22
18	2.12	Ship3	WepC	720	2.34
19	2.27	Ship2	WepB	814	2.54
20	2.43	Ship1	WepA	910	2.97
21	2.55	Ship4	WepC	661	2.77
22	2.70	Ship2	WepB	796	2.97
23	2.86	Ship1	WepA	854	3.47
24	2.98	Ship3	WepC	631	3.20
25	3.06	Ship1	WepE	403	3.28
26	3.21	Ship2	WepB	770	3.48
27	3.29	Ship3	WepE	389	3.51
28	3.45	Ship1	WepA	834	3.97
29	3.55	Ship3	WepD	431	3.67
30	3.70	Ship2	WepB	673	3.97
31	3.80	Ship1	WepD	407	3.92
32	3.96	Ship1	WepA	760	4.48
33	4.11	Ship2	WepB	585	4.38
34	4.23	Ship3	WepC	457	4.45
35	4.39	Ship1	WepA	740	4.98
36	4.54	Ship2	WepB	535	4.81
37	4.64	Ship2	WepD	210	4.76
38	4.79	Ship2	WepB	399	5.06
39	4.95	Ship1	WepA	713	5.48
40	5.10	Ship2	WepB	383	5.37
41	5.25	Ship2	WepE	310	5.62
42	5.41	Ship1	WepA	693	5.98
43	5.49	Ship2	WepE	124	5.71
44	5.64	Ship2	WepB	260	5.91
45	5.79	Ship2	WepB	164	6.16

46	5.95	Ship1	WepA	653	6.48
47	6.05	Ship1	WepD	120	6.17
48	6.20	Ship2	WepB	134	6.47
49	6.36	Ship1	WepA	625	6.98
50	6.51	Ship2	WepB	116	6.78
51	6.59	Ship1	WepE	106	6.81
52	6.74	Ship2	WepB	102	7.03
53	6.90	Ship1	WepA	605	7.48
54	7.02	Ship2	WepF	70	7.20
55	7.14	Ship2	WepF	59	7.38
56	7.32	Ship4	WepG	64	7.49
57	7.40	Ship1	WepA	575	7.98
58	7.50	Ship4	WepG	53	7.71
59	7.62	Ship4	WepG	38	7.87
60	7.83	Ship3	WepF	50	8.00
61	7.88	Ship1	WepA	555	8.48
62	8.00	Ship3	WepF	46	8.21
63	8.13	Ship3	WepF	40	8.36
64	8.24	Ship1	WepF	26	8.46
65	8.40	Ship1	WepA	519	8.98
66	8.55	Ship3	WepF	35	8.75
67	8.66	Ship3	WepF	28	8.90
68	8.80	Ship1	WepA	499	9.48
69	8.95	Ship3	WepF	24	9.15
70	9.06	Ship1	WepF	22	9.28
71	9.21	Ship3	WepF	20	9.41
72	9.32	Ship1	WepA	479	9.98
73	9.47	Ship3	WepF	16	9.67
74	9.58	Ship1	WepF	18	9.80
75	9.73	Ship	WepF	14	9.95
76	9.89	Ship1	WepA	451	10.48
77	10.04	Ship4	WepG	31	10.36
78	10.39	Ship1	WepA	427	10.98
79	10.54	Ship3	WepF	10	10.74
80	10.65	Ship1	WepF	7	10.86
81	10.80	Ship3	WepF	3	11.00
82	10.91	Ship1	WepA	372	11.48
83	11.06	Ship4	WepG	8	11.39
84	11.41	Ship1	WepA	352	11.98
85	11.56	Ship1	WepA	330	12.48
86	11.71	Ship2	WepF	44	12.22
87	12.37	Ship1	WepA	300	12.98
88	12.52	Ship1	WepA	280	13.48
89	12.67	Ship2	WepF	33	13.17
90	13.33	Ship1	WepA	250	13.98
91	13.48	Ship1	WepA	230	14.48
92	13.63	Ship2	WepF	12	14.13
93	14.29	Ship1	WepA	206	14.98
94	14.44	Ship1	WepA	184	15.48
95	14.59	Ship2	WepF	5	15.09
96	15.24	Ship1	WepA	154	15.98
97	15.39	Ship4	WepG	1	15.72
RTN:	15.72	EXPECTED COMBAT VALUE = 17871.80			

Table 18. 3-OPT VERTREP SCHEDULE - LOCAL NEIGHBORHOOD SEARCH

Sequence Number	Dispatch Time	Receiver	Weapon	Marginal Combat Value	Strike-down Compl.
1	.11	Ship4	WepE	858	.33
2	.21	Ship4	WepD	934	.33
3	.33	Ship4	WepC	1064	.55
4	.41	Ship3	WepE	804	.67
5	.53	Ship4	WepC	1002	.75
6	.63	Ship3	WepD	938	.75
7	.75	Ship3	WepC	870	.97
8	.91	Ship1	WepA	996	1.45
9	1.00	Ship1	WepE	774	1.22
10	1.09	Ship1	WepD	800	1.21
11	1.21	Ship3	WepC	864	1.43
12	1.37	Ship1	WepA	976	1.95
13	1.49	Ship3	WepC	786	1.71
14	1.61	Ship4	WepC	780	1.83
15	1.76	Ship2	WepB	880	2.03
16	1.92	Ship1	WepA	930	2.45
17	2.00	Ship4	WepE	523	2.22
18	2.12	Ship3	WepC	720	2.34
19	2.27	Ship2	WepB	814	2.54
20	2.43	Ship1	WepA	910	2.95
21	2.55	Ship4	WepC	661	2.77
22	2.70	Ship2	WepB	796	2.97
23	2.78	Ship1	WepE	403	3.00
24	2.94	Ship1	WepA	854	3.46
25	3.06	Ship3	WepC	631	3.28
26	3.21	Ship2	WepB	770	3.48
27	3.29	Ship3	WepE	389	3.51
28	3.45	Ship1	WepA	834	3.97
29	3.55	Ship3	WepD	431	3.67
30	3.70	Ship2	WepB	673	3.97
31	3.80	Ship1	WepD	407	3.92
32	3.96	Ship1	WepA	769	4.48
33	4.11	Ship2	WepB	585	4.38
34	4.23	Ship3	WepC	457	4.45
35	4.39	Ship1	WepA	740	4.98
36	4.54	Ship2	WepB	535	4.81
37	4.64	Ship2	WepD	210	4.76
38	4.79	Ship2	WepB	399	5.06
39	4.95	Ship1	WepA	713	5.48
40	5.10	Ship2	WepB	383	5.37
41	5.25	Ship2	WepB	310	5.62
42	5.41	Ship1	WepA	693	5.98
43	5.49	Ship2	WepE	124	5.71
44	5.64	Ship2	WepB	260	5.91
45	5.72	Ship1	WepE	106	5.94

46	5.88	Ship1	WepA	653	6.48
47	6.03	Ship2	WepB	164	6.30
48	6.13	Ship1	WepD	120	6.25
49	6.28	Ship2	WepB	134	6.55
50	6.44	Ship1	WepA	625	6.98
51	6.59	Ship2	WepB	116	6.86
52	6.71	Ship2	WepF	70	6.88
53	6.87	Ship1	WepA	605	7.48
54	7.02	Ship2	WepB	102	7.30
55	7.14	Ship2	WepF	59	7.38
56	7.32	Ship4	WepG	64	7.49
57	7.40	Ship1	WepA	575	7.98
58	7.50	Ship4	WepG	53	7.71
59	7.62	Ship4	WepG	38	7.87
60	7.83	Ship3	WepF	50	8.00
61	7.88	Ship1	WepA	555	8.48
62	8.00	Ship3	WepF	46	8.21
63	8.15	Ship3	WepF	40	8.36
64	8.24	Ship1	WepF	26	8.46
65	8.40	Ship1	WepA	519	8.98
66	8.55	Ship3	WepF	35	8.75
67	8.66	Ship3	WepF	28	8.90
68	8.80	Ship1	WepA	499	9.48
69	8.95	Ship3	WepF	24	9.15
70	9.06	Ship1	WepF	22	9.28
71	9.21	Ship3	WepF	20	9.41
72	9.32	Ship1	WepA	479	9.98
73	9.47	Ship3	WepF	16	9.67
74	9.58	Ship1	WepF	18	9.80
75	9.73	Ship1	WepF	14	9.95
76	9.89	Ship1	WepA	451	10.48
77	10.04	Ship4	WepG	31	10.36
78	10.39	Ship1	WepA	427	10.98
79	10.54	Ship3	WepF	10	10.74
80	10.65	Ship1	WepF	7	10.86
81	10.80	Ship1	WepA	372	11.48
82	10.95	Ship1	WepA	352	11.98
83	11.10	Ship2	WepF	44	11.60
84	11.76	Ship1	WepA	330	12.48
85	11.91	Ship1	WepA	300	12.98
86	12.06	Ship2	WepF	33	12.76
87	12.72	Ship3	WepF	3	12.91
88	12.83	Ship1	WepA	280	13.48
89	12.98	Ship4	WepG	8	13.30
90	13.33	Ship1	WepA	250	13.98
91	13.48	Ship1	WepA	230	14.48
92	13.63	Ship2	WepF	12	14.13
93	14.29	Ship1	WepA	206	14.98
94	14.44	Ship1	WepA	184	15.48
95	14.59	Ship2	WepF	5	15.09
96	15.24	Ship1	WepA	154	15.98
97	15.39	Ship4	WepG	1	15.72

RTN: 15.72 EXPECTED COMBAT VALUE = 17882.33

APPENDIX E. CONREP SCHEDULING DYNAMIC PROGRAM

A. PROGRAM LISTING

```
* -------------------------------------------------------
*             Variable Definitions
* -------------------------------------------------------
*   Index usage
*        J          Receivers
*        K          Lift number
*        R          Stages
*        S          States
*        IS         Ith state in stage
*        JS         Receivers in state S
*        RC         Stage complement
*        RM         Stage minus 1
*        SC         State complement
*        SM         State minus receiver
*   Coding of states and receivers (DATA statements)
*        JID(j)     Rcvr J Identity (binary code)
*        LRS(lj,s)  List of receivers in state s
*        SR(ls,r)   List of possible states in stage r
*        SRTOP(n)   Top state number with n receivers
*        TOPS       Top state number = SRTOP(n)
*   Input Data
*        NRCVR      Number of receivers
*        ETA        Expected attack time Ta
*        NL(j)      Total number of lifts req. by Rcvr j
*        CV(j,k)    Marginal C.V. of lift k on Rcvr j
*        X(j,k)     Transfer comp. time of lift k on Rcvr j
*        C(j,k)     Strikedown comp. time of lift k on Rcvr j
*   Derived Values
*        ATNR       Neg. recip. of Exp. attack time = -1 / ETA
*        CCV(j,k)   Cumulative CV of k lifts on Rcvr j
*        FBARX(j,k) Prob. X(j,k)>Ta
*   Stage Variables
*        FRS        Test Expected CV in state s at stage r
*        FOPT(r,s)  Optimal Expected CV in state s at stage r
*        JOPT(r,s)  ID of Optimal Rcvr in state s at stage r
*        KOPT(r,s)  Optimal Lifts to Rcvr JOPT(r,s)
*        XOPT(r,s)  Optimal Time allotted to JOPT(r,s)
*   Partition Variables
*        FP         Test Partition Expected CV
*        FPOPT      Opt. Partition Expected CV
*        R1OPT      Opt. Stage for Deliver side 1
*        R2OPT      Opt. Stage for Deliver side 2
*        S1OPT      Opt. State for Deliver side 1
*        S2OPT      Opt. State for Deliver side 2
* -------------------------------------------------------
```

```
*          Variable Declarations
* ----------------------------------------------------------
      INTEGER  J,R,S,JS,IS,K,RM,SM,RC,SC,NRCVR,NL(4),
     +   JID(4),SRTOP(4),TOPS,LRS(4,15),SR(6,4),NSR(4),
     +   JOPT(4,15),KOPT(4,15),R1OPT,R2OPT,S1OPT,S2OPT
      REAL  ETA,ATNR,CV(4,50),C(4,50),X(4,50),CCV(4,0:50),
     +   FBARX(4,50),FRS,FOPT(4,15),XOPT(4,15),FPOPT,FP
* ----------------------------------------------------------
*          Data statements
* ----------------------------------------------------------
* --------- initialization ---------------------------------
      DATA (CCV(J,0),J=1,4) /4*0./
      DATA R2OPT  /0/
      DATA S2OPT  /0/
* ------- coding of states and receivers -------------------
      DATA JID /1,2,4,8/
      DATA SRTOP /1,3,7,15/
      DATA NSR   /4,6,4,1/
* ----- list of possible states in each stage --------------
      DATA (SR(S,1),S=1,4) /1,2,4,8/
      DATA (SR(S,2),S=1,6) /3,5,6,9,10,12/
      DATA (SR(S,3),S=1,4) /7,11,13,14/
      DATA (SR(S,4),S=1,1) /15/
* ------- list of Receivers in each state ------------------
      DATA   LRS(1, 1)         /1/
      DATA   LRS(1, 2)         /2/
      DATA   LRS(1, 4)         /3/
      DATA   LRS(1, 8)         /4/
      DATA  (LRS(J, 3),J=1,2)  /2,1/
      DATA  (LRS(J, 5),J=1,2)  /3,1/
      DATA  (LRS(J, 6),J=1,2)  /3,2/
      DATA  (LRS(J, 9),J=1,2)  /4,1/
      DATA  (LRS(J,10),J=1,2)  /4,2/
      DATA  (LRS(J,12),J=1,2)  /4,3/
      DATA  (LRS(J, 7),J=1,3)  /3,2,1/
      DATA  (LRS(J,11),J=1,3)  /4,2,1/
      DATA  (LRS(J,13),J=1,3)  /4,3,1/
      DATA  (LRS(J,14),J=1,3)  /4,3,2/
      DATA  (LRS(J,15),J=1,4)  /4,3,2,1/
* ----------------------------------------------------------
*          Read data and initialize program
* ----------------------------------------------------------
* -------------------- Read Input Data ---------------------
      READ(5,*)NRCVR
         TOPS = SRTOP(NRCVR)
      READ(5,*)ETA
         ATNR = -1. / ETA
      DO 10 J=1,NRCVR
         READ(5,*)NL(J)
         DO 11 K=1,NL(J)
            READ(5,*)CV(J,K),X(J,K),C(J,K)
   11    CONTINUE
   10 CONTINUE
* -------------------- Compute Cumulative CV & FBARX ------
      DO 50 J=1,NRCVR
         DO 51 K=1,NL(J)
```

```
         CCV(J,K) =CCV(J,K-1)+CV(J,K)*EXP(C(J,K)*ATNR)
         FBARX(J,K)=EXP(X(J,K)*ATNR)
   51    CONTINUE
   50 CONTINUE
* ------------------------------------------------------------
*                    Single server DP
* ------------------------------------------------------------
* -------------------- Stage r=1 ------------------------
      DO 90 J=1,NRCVR
         FOPT(1,JID(J)) = CCV(J,NL(J))
         JOPT(1,JID(J)) = J
         KOPT(1,JID(J)) = NL(J)
         XOPT(1,JID(J)) = X(J,NL(J))
   90 CONTINUE
* -------------------- Stage r=2,NRCVR ------------------
      DO 100 R=2,NRCVR
         RM=R-1
*        *** for each state in this stage
         DO 110 IS=1,NSR(R)
            S=SR(IS,R)
            IF(S.GT.TOPS) GO TO 101
*           *** initialize maximization
            FOPT(R,S)=0.
            JOPT(R,S)=0
            KOPT(R,S)=0
            XOPT(R,S)=0.
*           *** for each receiver in state S
            DO 120 JS=1,R
               J=LRS(JS,S)
               SM=S-JID(J)
*              *** for each lift requested by rcvr J
               DO 130 K=1,NL(J)
                  FRS=CCV(J,K) + FOPT(RM,SM) * FBARX(J,K)
                  IF(FRS.GT.FOPT(R,S))THEN
                     FOPT(R,S)=FRS
                     JOPT(R,S)=J
                     KOPT(R,S)=K
                     XOPT(R,S)=X(J,K)
                  ENDIF
  130          CONTINUE
  120       CONTINUE
  110    CONTINUE
  101    CONTINUE
  100 CONTINUE
* ------------------------------------------------------------
*            Partition for two parallel servers
* ------------------------------------------------------------
* --------------------Initialize ------------------------
      R1OPT = NRCVR
      S1OPT = TOPS
      FPOPT = FOPT(R1OPT,S1OPT)
* -------------------- Stage r=1, INT( NRCVR/2 ) ---------
      DO 300 R=1,NRCVR/2
         RC = NRCVR - R
*        *** for each state in this stage
         DO 310 IS=1,NSR(R)
```

```
            S = SR(1S,R)
            IF(S.GT.TOPS) GO TO 301
            SC = TOPS - S
            FP = FOPT(R,S) + FOPT(RC,SC)
            IF(FP.GT.FPOPT)THEN
               FPOPT = FP
               R1OPT = R
               S1OPT = S
               R2OPT = RC
               S2OPT = SC
            ENDIF
  310    CONTINUE
  301    CONTINUE
  300 CONTINUE
* ---------------- Output schedule ----------------------
      WRITE(6,99901)
      WRITE(6,99905)
      S=S1OPT
      DO 210 R=R1OPT,1,-1
         WRITE(6,99910)JOPT(R,S),KOPT(R,S),XOPT(R,S)
         S=S-JID(JOPT(R,S))
  210 CONTINUE
      WRITE(6,99902)
      WRITE(6,99905)
      S=S2OPT
      DO 220 R=R2OPT,1,-1
         WRITE(6,99910)JOPT(R,S),KOPT(R,S),XOPT(R,S)
         S=S-JID(JOPT(R,S))
  220 CONTINUE
* -------------------------------------------------------
      STOP
* ------------------ formats ----------------------------
99901 FORMAT(' Delivery side 1: ')
99902 FORMAT(' Delivery side 2: ')
99905 FORMAT('   Receiver   Number of Lifts    Time Alongside')
99910 FORMAT(5X,I3,12X,I3,15X,F7.2)
* -------------------------------------------------------
      END
```

B. INPUT FILE

```
 4      NRCVR   Number of receivers
 4.     ETA     Estimate of expected time between raids
41      NL(1)   Number of lifts requested by receiver 1
    996.      0.16      0.66       Ship1     1     WepA
    976.      0.28      1.16       Ship1     2     WepA
    800.      0.40      0.50       Ship1     3     WepD
    774.      0.52      0.72       Ship1     4     WepE
    930.      0.64      1.66       Ship1     5     WepA
    910.      0.76      2.16       Ship1     6     WepA
    854.      0.88      2.66       Ship1     7     WepA
    834.      1.00      3.16       Ship1     8     WepA
    760.      1.12      3.66       Ship1     9     WepA
    407.      1.24      1.34       Ship1    10     WepD
    403.      1.36      1.56       Ship1    11     WepE
    740.      1.48      4.16       Ship1    12     WepA
    713.      1.60      4.66       Ship1    13     WepA
    693.      1.72      5.16       Ship1    14     WepA
    653.      1.84      5.66       Ship1    15     WepA
    625.      1.96      6.16       Ship1    16     WepA
    605.      2.08      6.66       Ship1    17     WepA
    575.      2.20      7.16       Ship1    18     WepA
    555.      2.32      7.66       Ship1    19     WepA
    519.      2.44      8.16       Ship1    20     WepA
    120.      2.56      2.66       Ship1    21     WepD
    499.      2.68      8.66       Ship1    22     WepA
    106.      2.80      3.00       Ship1    23     WepE
    479.      2.92      9.16       Ship1    24     WepA
    451.      3.04      9.66       Ship1    25     WepA
    427.      3.16     10.16       Ship1    26     WepA
    372.      3.28     10.66       Ship1    27     WepA
    352.      3.40     11.16       Ship1    28     WepA
    330.      3.52     11.66       Ship1    29     WepA
    300.      3.64     12.16       Ship1    30     WepA
    280.      3.76     12.66       Ship1    31     WepA
     26.      3.88      4.03       Ship1    32     WepF
    250.      4.00     13.16       Ship1    33     WepA
     22.      4.12      4.27       Ship1    34     WepF
    230.      4.24     13.66       Ship1    35     WepA
    206.      4.36     14.16       Ship1    36     WepA
     18.      4.48      4.63       Ship1    37     WepF
    184.      4.60     14.66       Ship1    38     WepA
     14.      4.72      4.87       Ship1    39     WepF
    154.      4.84     15.16       Ship1    40     WepA
      7.      4.96      5.11       Ship1    41     WepF
23      NL(2)   Number of lifts requested by receiver 2
    880.      0.16      0.41       Ship2     1     WepB
    814.      0.28      0.66       Ship2     2     WepB
    796.      0.40      0.91       Ship2     3     WepB
    770.      0.52      1.16       Ship2     4     WepB
    673.      0.64      1.41       Ship2     5     WepB
    585.      0.76      1.66       Ship2     6     WepB
    535.      0.88      1.91       Ship2     7     WepB
    399.      1.00      2.16       Ship2     8     WepB
    383.      1.12      2.41       Ship2     9     WepB
    310.      1.24      2.66       Ship2    10     WepB
```

210.	1.36	1.46	Ship2	11	WepD
260.	1.48	2.91	Ship2	12	WepB
124.	1.60	1.80	Ship2	13	WepE
164.	1.72	3.16	Ship2	14	WepB
134.	1.84	3.41	Ship2	15	WepB
116.	1.96	3.66	Ship2	16	WepB
70.	2.08	2.23	Ship2	17	WepF
102.	2.20	3.91	Ship2	18	WepB
59.	2.32	2.47	Ship2	19	WepF
44.	2.44	2.62	Ship2	20	WepF
33.	2.56	2.77	Ship2	21	WepF
12.	2.68	2.92	Ship2	22	WepF
5.	2.80	3.07	Ship2	23	WepF
20 NL(3)	Number of lifts requested by receiver 3				
938.	0.16	0.26	Ship3	1	WepD
870.	0.28	0.48	Ship3	2	WepC
864.	0.40	0.68	Ship3	3	WepC
804.	0.52	0.72	Ship3	4	WepE
786.	0.64	0.88	Ship3	5	WepC
720.	0.76	1.08	Ship3	6	WepC
631.	0.88	1.28	Ship3	7	WepC
431.	1.00	1.10	Ship3	8	WepD
457.	1.12	1.48	Ship3	9	WepC
389.	1.24	1.44	Ship3	10	WepE
50.	1.36	1.51	Ship3	11	WepF
46.	1.48	1.66	Ship3	12	WepF
40.	1.60	1.81	Ship3	13	WepF
35.	1.72	1.96	Ship3	14	WepF
28.	1.84	2.11	Ship3	15	WepF
24.	1.96	2.26	Ship3	16	WepF
20.	2.08	2.41	Ship3	17	WepF
16.	2.20	2.56	Ship3	18	WepF
10.	2.32	2.71	Ship3	19	WepF
3.	2.44	2.86	Ship3	20	WepF
13 NL(4)	Number of lifts requested by receiver 4				
1064.	0.16	0.36	Ship4	1	WepC
1002.	0.28	0.56	Ship4	2	WepC
934.	0.40	0.50	Ship4	3	WepD
858.	0.52	0.72	Ship4	4	WepE
780.	0.64	0.84	Ship4	5	WepC
661.	0.76	1.04	Ship4	6	WepC
523.	0.88	1.08	Ship4	7	WepE
64.	1.00	1.15	Ship4	8	WepG
53.	1.12	1.30	Ship4	9	WepG
38.	1.24	1.45	Ship4	10	WepG
31.	1.36	1.60	Ship4	11	WepG
8.	1.48	1.75	Ship4	12	WepG
1.	1.60	1.90	Ship4	13	WepG
CV(j,k)	X(j,k)	C(j,k)	ShipJ	K	
Marg. Combat Value	Transfer comp. time	Strike-down comp. time	Rcvr.	Lift No.	Weapon

APPENDIX F. DIFFUSION APPROXIMATION DIRECT STEADY-STATE SOLUTION

When the service discipline is probabilistic-longest-line, the diffusion approximation ordinary differential equations can be used to get a steady-state solution directly by setting the derivatives to zero; see Morrison, Gaver, and Pilnick [Ref. 42].

The method used to compute the steady-state mean involves setting the rate of change in the deterministic differential equations to zero, summing over all item types, using Newton's method to find the fixed point for the denominator of the $q_i(\mathbf{m}(t))$ terms, then backsolving for each steady-state $m_i(t)$ as follows:

From Equation (5.37)

$$\frac{dm_i(t)}{dt} = \lambda_i(\alpha_i - m_i(t)) - \mu_i \tilde{q}_i(\mathbf{m}(t)) \quad ;$$

for $i = 1, \ldots, I$. Setting the derivative to zero gives the steady-state condition

$$\lambda_i(\alpha_i - m_i(\infty)) = \mu_i \tilde{q}_i(\mathbf{m}(\infty)) \quad ;$$

for $i = 1, \ldots, I$. Using the PLL;i service discipline, this becomes

$$\lambda_i(\alpha_i - m_i(\infty)) = \mu_i \frac{\frac{w_i}{\mu_i} m_i(\infty)}{\sum_j \frac{w_j}{\mu_j} m_j(\infty)} \quad ;$$

for $i = 1, \ldots, I$. Letting

$$A = \sum_j \frac{w_j}{\mu_j} m_j(\infty) \quad ,$$

and solving for $m_i(\infty)$ gives

$$m_i(\infty) = \frac{A\,\lambda_i\,\alpha_i}{A\,\lambda_i + w_i} \quad ; \tag{F.1}$$

for $i = 1, \ldots, I$. Multiplying both sides by w_i/μ_i gives

$$\frac{w_i}{\mu_i}\, m_i(\infty) = \frac{A\,\lambda_i\,\alpha_i}{A\,\lambda_i + w_i}\;\frac{w_i}{\mu_i} \quad ;$$

for $i = 1, \ldots, I$. Summing over all i, the left hand side is then equal to A, which cancels giving

$$\sum_i \frac{\lambda_i\,\alpha_i\,w_i}{\mu_i(A\,\lambda_i + w_i)} \;-\; 1 \;=\; 0 \quad . \tag{F.2}$$

This expression is a function of only one variable, A, which may then be solved numerically by, for example, Newton's Method. The solution for A may then be used in (F.1) to solve for $m_i(\infty)$, $i = 1, \ldots, I$.

In the special case of unit weights and equal arrival rates for all items, a very long derivation obtains the following expression for the steady-state covariances, and hence, queue length variances:

$$\sigma_{ij}(\infty) = \frac{1}{2(\lambda A + 1)}\left[\frac{1}{A(2\lambda A + 1)}\left[\frac{m_i(\infty)\,m_j(\infty)\,B}{\lambda A} + \frac{A m_i(\infty)\,b_j^2}{\mu_j} + \frac{A m_j(\infty)\,b_i^2}{\mu_i}\right] + A\,b_i^2\,\delta_{ij}\right] \quad ;$$

for all i and j, where

$$A = \sum_j [m_j(\infty)/\mu_j] \quad ,$$

$$B = \sum_j [b_j^2/\mu_j^2] \quad .$$

and

$$b_i^2 = \lambda_i\,[\alpha_i - m_i(\infty)] + [m_i(\infty)/A] \quad .$$

Using his generating function approach for the steady-state under the same circumstances, Morrison gets the following solution for the covariances which more closely match the simulation results:

$$\sigma_{ij}(\infty) = \frac{m_i(\infty)\, m_j(\infty)}{(\lambda A + 1)(2\lambda A + 1)} \left[\frac{C}{A^3 \lambda} (2\lambda^2 A^2 + 2\lambda A + 1) - \lambda \left(\frac{1}{\mu_i} + \frac{1}{\mu_j} \right) \right] + \frac{m_i(\infty)\, \delta_{ij}}{\lambda A + 1} \quad ;$$

for all i and j, where

$$A = \sum_j [m_j(\infty)/\mu_j] \quad ,$$

and

$$C = \sum_j [m_j(\infty)/\mu_j^2] \quad .$$

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